Large-N expansion for the 2D Dyson gas

Large-*N* expansion for the 2D Dyson gas

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Abstract
We discuss the $1/N$ expansion of the free energy of N logarithmically
interacting charges in the plane in an external field. For some particular
values of the inverse temperature β, this system is equivalent to the eigenvalue
version of certain random matrix models, where it is referred to as the 'Dyson
gas' of eigenvalues. To find the free energy at large N and the structure of
$1/N$-corrections, we first use the effective action approach and then confirm
the results by solving the loop equation. The results obtained give some new
representations of the mathematical objects related to the Dirichlet boundary
value problem, complex analysis and spectral geometry of exterior domains.
They also suggest interesting links with bosonic field theory on Riemann
surfaces, gravitational anomalies and topological field theories.

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1. Introduction

In this paper we discuss the large-*N* expansion of the *N*-fold integral

$$Z_N = \int |\Delta_N(z_i)|^{2\beta} \prod_{j=1}^{N} e^{\frac{1}{\hbar} W(z_j)} \, d^2 z_j. \tag{1.1}$$

Here W is a function of $z = x + iy$ and $\bar{z} = x - iy$, $\beta > 0$ is a parameter, $\Delta_N(z_i) = \prod_{i>j}^{N} (z_i - z_j)$ is the Vandermonde determinant and $d^2z \equiv dx\, dy$. The 'Planck constant' $\hbar$ is introduced here to stress the quasiclassical nature of the large N limit: $N \to \infty$ together with $\hbar \to 0$ so that $t_0 = N\hbar$ is kept fixed.

The integral is equal to the partition function of the statistical ensemble of N two-dimensional (2D) Coulomb charges in the external potential W (the 'Dyson gas' [1]). Different aspects of 2D Coulomb plasma were discussed in [2–7]; for further applications, see [8–12]. In this interpretation, the parameter β is the inverse temperature. At large β the Dyson gas

is believed to form a Wigner crystal. In this paper we assume, however, that β is such that the system is in the liquid phase as $N \to \infty$. At some particular values of β the model can be viewed as the eigenvalue version of certain ensembles of random matrices [13] (normal or complex matrices at $\beta = 1$ [14, 15] and normal self-dual matrices at $\beta = 2$).

An important piece of information is encoded in the $1/N$-expansion of the model. For random matrices, it has the meaning of the 'genus expansion' since the $1/N$ order of perturbation theory graphs is determined by their Euler characteristics. In the following, we prefer to work with the equivalent $\hbar$-expansion, thus emphasizing its semiclassical nature:

$$\log Z_N = c(N) + \frac{F_0}{\hbar^2} + \frac{F_{1/2}}{\hbar} + F_1 + O(\hbar). \tag{1.2}$$

The explicit form of the $c(N)$ is given below. It can be absorbed by normalization. With some abuse of terminology we call $\hbar^2 \log Z_N$ a free energy.

When N becomes large new macroscopic structures emerge. The gas segregates into 'phases' with zero and nonzero density separated by a very narrow interface. Let D be the domain in the complex plane where the density is nonzero (it may consist of several disconnected components). In the first approximation, the gas looks like a continuous charged fluid trapped in the domain D. The density at any point outside it is exponentially small as $N \to \infty$.

The first two terms in (1.2), F_0 and $F_{1/2}$, are of purely classical nature in the sense that only the static equilibrium state of the charges (the saddle point of the integral) contributes to them. The leading contribution to the free energy, F_0, is basically the Coulomb energy of the charged fluid in the domain D. Taking into account the discrete àtomic' structure of the Dyson gas, which implies a short-distance cutoff and entropy of macroscopic states, one is able to find the correction $F_{1/2}$ to the 'classical' free energy. The next term, F_1, apart from further corrections of the classical nature, includes contribution from small fluctuations about the equilibrium state.

The $\beta = 1$ Dyson gas confined to the line is related to the model of Hermitian random matrices. The $1/N$-expansion of this model beyond the leading order has been obtained in the seminal paper [16]. Recently, there was a progress in understanding these results from the algebro-geometric point of view [17–19] and in extending them to other matrix models [20–22]. In [23], the genus-1 correction was interpreted in terms of free bosons on Riemann surfaces. Our results for F_1 (partially reported in [22]) enjoy a similar interpretation and lead to interesting connections with spectral geometry of planar domains. In particular, our results suggest a formula for the determinant of the Laplace operator in exterior planar domains in terms of the conformal map of the domain onto the exterior of the unit circle. For polynomial potentials, F_1 enjoys a finite determinant representation (6.9) similar to the one known in topological field theories.

Our results for F_1 suggest a new deep connection between the 2D Dyson gas and 2D quantum gravity. This connection does not explore the well-known approach to random surfaces through a scaling limit of random matrices. It rather indicates that the density of 2D Dyson particles can be treated as a fluctuating 2D metric. We do not develop this approach in this paper.

In the rest of the introductory section we fix the notation and present some standard exact relations to be used in the sequel. We follow [22, 24, 25].

The main observables in the Dyson gas statistical ensemble are mean values and correlators of symmetric functions of the particles coordinates. Let $A(z_1, \ldots, z_N)$ be such a function, then the mean value $\langle A \rangle$ is defined by

$$\langle A \rangle = \frac{1}{Z_N} \int |\Delta_N(z_i)|^{2\beta} A(z_1, \ldots, z_N) \prod_{j=1}^{N} e^{\frac{1}{\hbar} W(z_j)} \, d^2 z_j.$$

A particularly important example is the density

$$\rho(z) = \hbar \sum_j \delta(z - z_j), \tag{1.3}$$

where $\delta(z)$ is the two-dimensional δ-function. Instead of correlations of density, it is often convenient to consider correlations of the field,

$$\varphi(z) = -\beta \int \log |z - \zeta|^2 \rho(\zeta) \, d^2 \zeta, \tag{1.4}$$

from which the correlations of density can be found by means of the relation

$$4\pi \beta \rho(z) = -\Delta \varphi(z). \tag{1.5}$$

Here and below, $\Delta = 4 \partial_z \partial_{\bar{z}}$ is the Laplace operator. Clearly, φ is the 2D Coulomb potential created by the charges.

Handling with multipoint correlation functions, it is customary to pass to their connected parts. For example, in the case of 2-point functions, the connected correlation function is defined as

$$\langle \rho(z_1) \rho(z_2) \rangle_c \equiv \langle \rho(z_1) \rho(z_2) \rangle - \langle \rho(z_1) \rangle \langle \rho(z_2) \rangle.$$

The following variational formulae hold true:

$$\langle \rho(z) \rangle = \hbar^2 \frac{\delta \log Z_N}{\delta W(z)}, \qquad \langle \rho(z_1) \rho(z_2) \rangle_c = \hbar^2 \frac{\delta \langle \rho(z_1) \rangle}{\delta W(z_2)} = \hbar^4 \frac{\delta^2 \log Z_N}{\delta W(z_1) \delta W(z_2)}. \tag{1.6}$$

These formulae are exact for any finite N. They follow from the fact that variation of the partition function over a general potential W inserts $\sum_i \delta(z - z_i)$ into the integral. More generally, the connected part of the $(n + 1)$-point density correlation function is given by the linear response of the n-point one to a small variation of the potential.

Let $f = f(z, \bar{z})$ be a function in the complex plane (for brevity we write simply $f(z)$ in what follows). Summing over the charges, we get the symmetric function $\sum_i f(z_i) \equiv \text{Tr} f$, where the notation is inspired by related models of random matrices. Mean values and correlators of such functions are expressed through those of densities:

$$\langle \text{Tr} f \rangle = \int \langle \rho(z) \rangle f(z) \, d^2 z, \quad \langle \text{Tr} f_1 \, \text{Tr} f_2 \rangle_c = \int \langle \rho(z_1) \rho(z_2) \rangle_c f_1(z_1) f_2(z_2) \, d^2 z_1 \, d^2 z_2 \tag{1.7}$$

and so on.

2. Ward identities

Clearly, the integral (1.1) remains the same if we change the integration variables. In other words, it is invariant under reparametrizations of the z-coordinate. This leads to a number of Ward identities which are our basic tool to compute the free energy at large N.

2.1. Holomorphic form of the Ward identity at finite N: loop equation

We begin with a holomorphic reparametrization $z_i \to z_i + \epsilon(z_i)$. Let us apply it to the integral (1.1), $Z_N = \int e^{-\frac{1}{\hbar} E(z_1, \ldots, z_N)} \prod_j d^2 z_j$, where the energy is

$$-\hbar^{-1} E = \beta \sum_{i \neq j} \log |z_i - z_j| + \hbar^{-1} \sum_j W(z_j). \tag{2.1}$$

In the first order the integrand transforms as

$$E \longrightarrow E + \sum_l \left(\frac{\partial E}{\partial z_l} \epsilon(z_i) + \frac{\partial E}{\partial \bar{z}_l} \overline{\epsilon(z_i)} \right),$$

while the volume element $\prod_j \mathrm{d}^2 z_j$ undergoes the scaling (Weyl) transformation

$$\prod_j \mathrm{d}^2 z_j \longrightarrow \left[1 + \sum_l (\partial \epsilon(z_l) + \overline{\partial \epsilon(z_l)}) \right] \prod_j \mathrm{d}^2 z_j.$$

The invariance of the integral is then expressed by the identity

$$\sum_i \int \frac{\partial}{\partial z_i} \left(\epsilon(z_i) \, \mathrm{e}^{-\frac{1}{\hbar} E} \right) \prod_j \mathrm{d}^2 z_j = 0$$

valid for any ϵ. Introducing a suitable cutoff at infinity, if necessary, one sees that the 2D integral over z_i can be transformed, by virtue of the Green theorem, into a contour integral around infinity and so it does vanish.

Let us take

$$\epsilon(z_i) = \frac{\epsilon}{z - z_i}, \tag{2.2}$$

where z is a complex parameter. The singularity at the point z does not destroy the above identity since its contribution is proportional to the vanishing integral $\oint \mathrm{d}\bar{z}_i / (z_i - z)$ over a small contour encircling z. We explore this singularity in the following. Therefore, we have the equality

$$\sum_i \int \left[-\frac{\partial_{z_i} E}{z - z_i} + \frac{\hbar}{(z - z_i)^2} \right] \mathrm{e}^{-\frac{1}{\hbar} E} \prod_j \mathrm{d}^2 z_j = 0,$$

where $\hbar^{-1} \partial_{z_i} E = -\beta \sum_{l \neq i} \frac{1}{z_i - z_l} - \hbar^{-1} \partial W(z_i)$. Using the identity

$$\sum_{i,j} \frac{1}{(z - z_i)(z - z_j)} = \sum_{i \neq j} \frac{2}{(z - z_i)(z_i - z_j)} + \sum_i \frac{1}{(z - z_i)^2},$$

we rewrite it in the form

$$\langle \mathcal{T} \rangle = 0, \tag{2.3}$$

where we define the holomorphic component of the stress energy tensor

$$-2\beta \mathcal{T} = \frac{2}{\hbar} \sum_i \frac{\partial W(z_i)}{z - z_i} + \beta \left(\sum_i \frac{1}{z - z_i} \right)^2 + (2 - \beta) \sum_i \frac{1}{(z - z_i)^2}. \tag{2.4}$$

This is the holomorphic form of the Ward identity. In terms of the field $\varphi(z)$ (1.4) it reads

$$\frac{1}{2\pi} \int \frac{\partial W(\zeta) \langle \Delta \varphi(\zeta) \rangle}{z - \zeta} \, \mathrm{d}^2 \zeta = \langle (\partial \varphi(z))^2 \rangle + (2 - \beta) \hbar \langle \partial^2 \varphi(z) \rangle. \tag{2.5}$$

The correlator at coinciding points is understood as $\langle (\partial \varphi(z))^2 \rangle = \lim_{z' \to z} \langle \partial \varphi(z) \partial \varphi(z') \rangle$.

We have got an exact relation between 1- and 2-point correlation functions, valid for any finite N. For historical reasons, it is called the *loop equation*. Since correlation functions are variational derivatives of the free energy, the loop equation is an implicit functional relation for the free energy. However, it is not a closed relation. It can be made closed by some additional assumptions or approximations. A combination with the $1/N$-expansion is particularly meaningful.

2.2. Path integral representation of the partition function and the Weyl form of the Ward identity

Here we give somewhat heuristic but transparent arguments to calculate the free energy by using the path integral representation and the Weyl form of the Ward identity. The results will be justified and refined by means of the loop equation.

Density as a metric. At large N, when the density can be treated as a smooth function, the partition function of the Dyson gas can be represented as a path integral over densities $\rho(z)$. Symbolically, we write the partition function as

$$Z_N = \int [D\rho] \, e^{-\frac{1}{\hbar^2} \mathcal{A}[\rho]}, \tag{2.6}$$

where the action $\mathcal{A}$ is to be determined. In this approach $\rho(z) \, dz \, d\bar{z}$ appears as a metric $g_{ab} \, dx^a \, dx^b$ written in the conformal gauge $g_{ab} = \delta_{ab}\rho(z)$, while the loop equation has the meaning of the Ward identity with respect to holomorphic diffeomorphisms.

Stress energy tensor. The action in equation (2.6) can be determined by its response to a change of the metric. The response of the action to variation of the metric is generated by the stress energy tensor (s.e.t.). Under the Weyl transformation $\rho \to \rho + \delta\rho$, the change of the action is

$$-\delta\mathcal{A} = \frac{1}{\pi} \int \mathcal{T}_{z\bar{z}} \delta\rho \rho^{-1} \, d^2z \tag{2.7}$$

where $\mathcal{T}_{z\bar{z}}$ is the trace of the s.e.t. The easiest way to determine $\mathcal{T}_{z\bar{z}}$ is to use the conservation law of the s.e.t. reflecting the reparametrization invariance of the action. In the conformal gauge the conservation law reads

$$\bar{\partial}\mathcal{T} + \rho\partial(\rho^{-1}\mathcal{T}_{z\bar{z}}) = 0, \tag{2.8}$$

where $\mathcal{T}$ is the holomorphic component of the s.e.t. It has already been derived directly from the finite dimensional integral (1.1). The result (equation (2.4)) can be rewritten as

$$-2\beta\mathcal{T}(z) = 2\beta \int \frac{\partial W(\zeta)}{z - \zeta} \rho(\zeta) \, d^2\zeta + (\partial\varphi(z))^2 + (2 - \beta)\hbar\partial^2\varphi(z). \tag{2.9}$$

Let us apply $\bar{\partial}$ to this equality. Taking into account that $\bar{\partial}(1/z) = \pi\delta(z)$ and using (1.5), we obtain $\bar{\partial}\mathcal{T} = \pi\rho \left(\partial\varphi - \partial W + \frac{\hbar}{2}(2 - \beta)\partial\log\rho\right)$. The conservation law then states that

$$\mathcal{T}_{z\bar{z}} = \pi\rho \left[W - \varphi - \tfrac{1}{2}(2 - \beta)\hbar\log(e^{\lambda+1}\rho) \right], \tag{2.10}$$

where λ is the integration constant which will be determined by the normalization condition $\int \langle\rho\rangle \, d^2z = \hbar N$.

The action. Integrating equation (2.7) with $\mathcal{T}_{z\bar{z}}$ given by (2.10), one determines the action[5] up to a constant:

$$-\mathcal{A} = \frac{1}{8\pi} \int (\beta^{-1}\varphi(\rho^{-1}\Delta)\varphi + 8\pi W - 4\pi(2 - \beta)\hbar\log(e^\lambda\rho))\rho \, d^2z. \tag{2.11}$$

Here $\rho^{-1}\Delta$ is the invariant Laplace–Beltrami operator in the metric $\rho \, dz \, d\bar{z}$.

Let us discuss the physical meaning of this action. It consists of energy and entropy contributions. The energy is given by (2.1). On the scales much larger than the mean distance between the charges the system can be treated as a charged liquid with the electrostatic energy

$$-\hbar E_0[\rho] = \beta \iint \rho(z)\log|z - \zeta|\rho(z') \, d^2z \, d^2z' + \int W(z)\rho(z) \, d^2z$$

$$= \frac{1}{8\pi} \int (\beta^{-1}\varphi(\rho^{-1}\Delta)\varphi + 8\pi W)\rho \, d^2z. \tag{2.12}$$

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[5] The effective action for some other models of random matrices has been discussed in [26, 27].

It gives the leading term of the action. The correction $\frac{\hbar}{2}(2 - \beta) \int \rho \log \rho \, d^2z$ results from the discrete 'atomic' structure of the Dyson gas. The argument below goes back to Dyson [1].

The subleading term of the action consists of two contributions of different nature. One correction comes from the sum $\sum_{i \neq j} \log|z_i - z_j|$, when passing to the continuous theory. Namely, one should exclude the terms with $i = j$, writing

$$\sum_{i \neq j} \log|z_i - z_j| = \sum_{i,j} \log|z_i - z_j| - \sum_j \log \ell(z_j)$$

or

$$\sum_{i \neq j} \log|z_i - z_j| = \hbar^{-2} \int \rho(z) \log|z - z'| \rho(z') \, d^2z \, d^2z' - \hbar^{-1} \int \rho(z) \log \ell(z) \, d^2z,$$

where ℓ is a short-distance cutoff (which may depend on the point z_j). It is natural to take the cutoff to be

$$\ell(z) \simeq \sqrt{\frac{\hbar}{\rho(z)}}, \tag{2.13}$$

which is the mean distance between the charges around the point z. This gives the improved estimate for the electrostatic energy:

$$E[\rho] = E_0[\rho] - \frac{1}{2}\beta \int \rho \log \rho \, d^2z + \frac{1}{2}\beta\hbar N \log \hbar - \gamma_1 \hbar N, \tag{2.14}$$

where γ_1 is a numerical constant which cannot be determined by this argument.

Another correction comes from the integration measure when one passes from the integration over z_j to the integration over macroscopic densities. We can write

$$\prod_j d^2z_j = N! J[\rho][D\rho],$$

where $[D\rho]$ is an integration measure in the space of densities, $J[\rho]$ is the Jacobian of this change of variables and the factor $N!$ takes into account the symmetry under permutations (all the states that differ by a permutation of the charges are identical). To estimate the Jacobian, we divide the plane into N microscopic 'cells' such that jth particle occupies a cell of size $\ell(z_j)$, where $\ell(z_j)$ is the mean distance (2.13) between the particles around the point z_j. All the microscopic states in which the particles remain in their cells are macroscopically indistinguishable. Given a macroscopic density ρ, $J[\rho]$ is then approximately equal to the integral $\int_{cells} \prod_j d^2z_j$, with each particle being confined to its own cell. Therefore, $J[\rho] \sim \prod_j \ell^2(z_j)$, and thus $\log J[\rho]$ (sometimes referred to as entropy of the state with the macroscopic density ρ) is given by

$$\hbar \log J[\rho] = -\int \rho \log \rho \, d^2z + \hbar N \log \hbar + \gamma_2 \hbar N, \tag{2.15}$$

where γ_2 is another numerical constant[6].

The subleading term of the action, $-\frac{1}{2}(2 - \beta) \int \rho \log \rho \, d^2z$, is thus the sum of the contribution due to the short distance cutoff and the entropy contribution. They cancel each other in the ensemble of normal self-dual matrices (at $\beta = 2$).

The Weyl form of the loop equation. The gravitational Ward identities state that the expectation value of the variation of the action vanishes. They can be written in two complimentary (holomorphic and Weyl) forms:

$$\langle \mathcal{T} \rangle = 0, \qquad \langle \mathcal{T}_{z\bar{z}} \rangle = 0. \tag{2.16}$$

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[6] Combining (2.14), (2.15) and taking into account the factor $N!$ in the measure, we obtain for the $c(N)$ in equation (1.2): $c(N) = \log N! + \frac{N}{2}(2 - \beta) \log \hbar + \gamma N$, where $\gamma = \gamma_1 + \gamma_2$ is a numerical constant.

The first identity, generated by the holomorphic component of the s.e.t., is the holomorphic loop equation (2.5). The second one, $-\frac{1}{\pi}\langle T_{z\bar{z}}\rangle = \langle \rho \frac{\delta \mathcal{A}}{\delta \rho}\rangle = 0$, i.e.,

$$\langle \varphi \rho \rangle - W\langle \rho \rangle + \tfrac{1}{2}(2-\beta)\hbar \langle \rho \log(e^{\lambda+1}\rho)\rangle = 0, \tag{2.17}$$

is another form of the loop equation.

The Weyl form of the Ward identity represents the gravitational anomaly. It is based on the fact that the diffeomorphism (2.2) is not holomorphic at the points z_i (positions of the particles). At these points $\bar{\partial}\epsilon(z) = \epsilon \pi \delta(z-z_i)$ and the holomorphic component $\mathcal{T}$ of the s.e.t. is not analytic. It has simple and double poles.

3. The structure of the $1/N$ expansion

The large N limit we are interested in ($N\hbar = t_0$ remains finite) corresponds to a very low effective temperature of the gas, when fluctuations around equilibrium positions of the charges are negligible. The main contribution to the partition function then comes from a configuration, where the charges are 'frozen' at their equilibrium positions and, moreover, the gas can be treated as a continuous fluid at static equilibrium. Mathematically, all this means that the integral (1.1) is evaluated by the saddle point method, with only the leading contribution being taken into account. Fluctuations around the saddle point give $1/N$ corrections.

The path integral (2.6) makes the structure of the $1/N$ expansion intuitively clear. First we find an equilibrium ('classical') density ρ_{cl} which minimizes the action $\mathcal{A}[\rho]$. Then, separating the classical part of the density, $\rho = \rho_{cl} + \hbar\delta\rho$ we can write

$$\mathcal{A}[\rho] = \mathcal{A}[\rho_{cl}] + \frac{\hbar^2}{2}\int \delta\rho(z)K(z,z')\delta\rho(z')\,d^2z + \cdots$$

where

$$K(z,z') = \frac{\delta^2\mathcal{A}[\rho]}{\delta\rho(z)\delta\rho(z')}\bigg|_{\rho=\rho_{cl}}$$

is the kernel of an integral operator $\hat{K}$. The path integral representation of the Dyson gas immediately produces the first two leading contributions to the free energy:

$$\hbar^2 \log Z_N = -\mathcal{A}[\rho_{cl}] - \frac{\hbar^2}{2}\log \det \hat{K} + O(\hbar^3). \tag{3.1}$$

The first term is the classical value of the action, the second one is due to the Gaussian fluctuations around the classical solution[7].

Since the leading part of the free energy is the classical value of the action, the connected part of the pair density correlation computed in the leading order is equal to the kernel of the operator inverse to $\hat{K}$:

$$\hbar^{-2}\langle \rho(z)\rho(z')\rangle_c = \langle \delta\rho(z)\delta\rho(z')\rangle = \hat{K}^{-1}(z,z') \qquad (\hbar \to 0). \tag{3.2}$$

The term coming from fluctuations (the second term in (3.1)) is not the only contribution to F_1. The action itself and the solution φ_{cl} of the equation

$$W - \varphi_{cl} = \frac{1}{2}(2-\beta)\hbar \log\left(-\frac{e^{\lambda+1}}{4\pi\beta}\Delta\varphi_{cl}\right) \tag{3.3}$$

minimizing the action depend on $\hbar$. Expanding the density, $\rho_{cl} = \rho_0 + \hbar\rho_{1/2} + \cdots$, we obtain

$$\mathcal{A}[\rho_{cl}] = \mathcal{A}[\rho_0] - \frac{\hbar^2}{2}\int \rho_{1/2}(z)K(z,z')\rho_{1/2}(z')\,d^2z\,d^2z' + \cdots. \tag{3.4}$$

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[7] One can obtain these results by a direct iteration of the Ward identity (2.17) without appealing to the path integral representation.

The action evaluated at ρ_0 yields the first two leading terms of the free energy. Up to a constant we write

$$-\mathcal{A}[\rho_0] = F_0 + \hbar F_{1/2}.$$

The second term in (3.4) contributes to the next order:

$$-\mathcal{A}[\rho_{\text{cl}}] + \mathcal{A}[\rho_0] = \hbar^2 F_1^{(1)}.$$

Summing up, we obtain three leading orders of the $1/N$ expansion (1.2):

$$F_0 = -\beta \iint \rho_0(z) \log \left| \frac{1}{z} - \frac{1}{\zeta} \right| \rho_0(\zeta) \, \mathrm{d}^2 z \, \mathrm{d}^2 \zeta \tag{3.5}$$

$$F_{1/2} = -\frac{1}{2}(2 - \beta) \int \rho_0 \log \rho_0 \, \mathrm{d}^2 z \tag{3.6}$$

$$F_1 = F_1^{(0)} + F_1^{(1)} \tag{3.7}$$

$$F_1^{(0)} = -\frac{1}{2} \log \det \hat{K}, \tag{3.8}$$

$$F_1^{(1)} = \frac{1}{2} \int \rho_{1/2}(z) K(z, z') \rho_{1/2}(z') \, \mathrm{d}^2 z \, \mathrm{d}^2 z'. \tag{3.9}$$

They are expressed entirely through the mean density and the pair correlation function (3.2) computed in the leading order. This remains to be the case for the higher order corrections as well.

The mean density and the pair correlation function were found in [24]. Using these results one is able to compute the free energy by means of (3.5)–(3.7). We summarize the results in the next section. In the following we derive them on a more solid basis by a direct iteration of the holomorphic loop equation.

We note also that the $\hbar$-expansion of the free energy can be written in the 'topological' form $\sum_{g \geqslant 0} \hbar^{2g} \tilde{F}_g$, where each term has its own expansion in $\varepsilon = (2 - \beta)\hbar$: $\tilde{F}_g = \tilde{F}_g^{(0)} + \sum_{n \geqslant 1} \varepsilon^n \tilde{F}_g^{(n)}$.

4. The main results

We introduce the following notation:

$$\sigma(z) = -\frac{1}{4\pi} \Delta W(z), \qquad \chi(z) = \log \sqrt{\pi \sigma(z)}, \qquad \alpha = \sqrt{\frac{2}{\beta}} - \sqrt{\frac{\beta}{2}}. \tag{4.1}$$

4.1. Summary of the results

In order to compute the first three leading contributions to the free energy we need the following results.

- *The mean density* computed as a power expansion in $\hbar$ vanishes outside a bounded domain D (we assume that D is connected). Inside the domain the first two orders are

$$\rho_0 = \beta^{-1} \sigma, \qquad z \in \mathsf{D}; \tag{4.2}$$

$$\rho_{1/2} = -(2 - \beta) \hat{K}^{-1} \chi \qquad z \in \mathsf{D}. \tag{4.3}$$

The leading correction to the density $\rho_{1/2}$ is singular at the boundary (see equation (5.16)). Corrections exponential in $\hbar$ make the density a smooth function falling exponentially outside D.

- *The shape of* D is determined by the potential W and the number of particles, as described below (see (4.16)). It is the subject of the inverse potential problem in 2D.
- *The pair correlation function of densities* is related to the Dirichlet boundary problem in the complementary domain $\mathbb{C}\backslash D$ (the exterior of D). Given a function $f(z)$, let $f^H(z)$ be its harmonic continuation from the boundary to the exterior of D (the solution of the exterior Dirichlet boundary value problem). The pair correlation function of densities (and thus the kernel $K(z_1, z_2)$) is completely characterized by the integral relation

$$4\pi\beta \int f(z)\hat{K}^{-1}(z, z')g(z')\,\mathrm{d}^2z\,\mathrm{d}^2z'$$
$$= -\int_{\mathbb{C}\backslash D} \nabla(f - f^H)\nabla(g - g^H)\,\mathrm{d}^2z + \int_{\mathbb{C}} \nabla f \nabla g\,\mathrm{d}^2z \tag{4.4}$$

for any smooth functions f, g. Hereafter, $\hat{K}^{-1}(z, z')$ is the kernel of the operator $\hat{K}^{-1}$. The first integral goes over the exterior of the domain D while the second one is over the entire plane. Note that $f - f^H$ vanishes on the boundary.
- *The spectral determinant* $\det \hat{K}$ in (3.8) is a ratio of two spectral determinants of the invariant Laplace–Beltrami operators $e^{-2\chi}\Delta$ in the conformal metric $\pi\sigma = e^{2\chi}$:

$$\log \det \hat{K} = \log \frac{\det(-e^{-2\chi}\Delta_{\mathbb{C}\backslash D})}{\det(-e^{-2\chi}\Delta_{\mathbb{C}})}. \tag{4.5}$$

One of them acts on the entire plane (and so does not depend on D). The other one is the Laplace–Beltrami operator in the *exterior domain* $\mathbb{C}\backslash D$ acting on functions vanishing at the boundary. The determinants of the Laplacians in exterior, unbounded domains were introduced in [28]. The definition is more involved than the usual zeta-function regularization. For the details, see [28] and references therein. However, explicit formulae for the exterior determinants were not known. Our result for $F_1^{(0)}$ obtained by a direct solution of the iterated loop equation suggests such a formula (see below).

To present the results for the free energy we need some more notation. Let $\mathrm{d}s \equiv |\mathrm{d}z|$ be the line element along the boundary of D, κ be the local curvature of the boundary, and $e^\phi\,\mathrm{d}w\,\mathrm{d}\bar{w} = \mathrm{d}z\,\mathrm{d}\bar{z}$ be the conformal metric induced by the conformal map $w(z)$ of the exterior of D onto the exterior of the unit circle.

- *Free energy.*

$$F_0 = -\frac{1}{\beta} \int_D \int_D \sigma(z) \log \left| \frac{1}{z} - \frac{1}{\zeta} \right| \sigma(\zeta)\,\mathrm{d}^2z\,\mathrm{d}^2\zeta \tag{4.6}$$

$$F_{1/2} = -\frac{2-\beta}{2\beta} \int_D \sigma \log(\pi\sigma)\,\mathrm{d}^2z \tag{4.7}$$

$$F_1^{(0)} = \frac{1}{24\pi} \left[\int_{|w|>1} |\nabla(\phi + \chi)|^2\,\mathrm{d}^2w - 2 \oint_{|w|=1} (\phi + \chi)|\mathrm{d}w| \right]$$
$$- \frac{1}{24\pi} \int_{\mathbb{C}} |\nabla\chi|^2\,\mathrm{d}^2z - \frac{1}{8\pi} \oint_{\partial D} \partial_n \chi\,\mathrm{d}s \tag{4.8}$$

$$F_1^{(1)} = -\frac{\alpha^2}{4\pi} \left[\int_{\mathbb{C}\backslash D} |\nabla(\chi - \chi^H)|^2\,\mathrm{d}^2z - \int_{\mathbb{C}} |\nabla\chi|^2\,\mathrm{d}^2z \right] + \frac{\mu + \frac{1}{4}}{2\pi} \oint_{\partial D} \partial_n \chi\,\mathrm{d}s. \tag{4.9}$$

Note that $F_{1/2}$ here differs from (3.6) by a constant times N which is a matter of normalization. We have determined the coefficient μ only at $\beta = 1$. In this case $\mu = -1/8$.

- *Special cases.*

    (a) A quasiharmonic potential $W = -|z|^2 + V(z) + \overline{V(z)}$. The density is uniform inside the domain, $\sigma = 1/\pi$, $F_{1/2} = F_1^{(1)} = 0$,

    $$F_1 = -\frac{1}{24\pi} \oint_{|w|=1} (\phi\partial_n\phi + 2\phi)|dw| \tag{4.10}$$

    (b) $\beta = 2$. In this case the 'classical' corrections ($F_{1/2}$ and $F_1^{(1)}$) vanish for any potential W. $F_1^{(0)}$ is given by (4.8).

These results, being combined with the spectral determinant representation (3.8), (4.5), suggest an explicit formula for the determinant of the Laplace–Beltrami operator in exterior domains. For the interior domains, such a formula has been known due to [29, 30]. It is called the Polyakov–Alvarez formula. For simply connected domains, it can be written in terms of the conformal map of the domain onto the unit disc. (A sketch of the derivation is given in appendix D.) Our results suggest that for the exterior case this formula remains basically the same, with the interior conformal map being substituted by the exterior one. Below we list some formulae of this type. Their rigorous proof is a challenging problem which is beyond the scope of this paper.

- *The Polyakov–Alvarez formula for exterior determinants (a conjecture).*

    $$\log\frac{\det(-e^{-2\chi}\Delta_{\mathbb{C}\backslash D})}{\det(-\Delta_{\mathbb{C}\backslash D})} = -\frac{1}{12\pi}\int_{\mathbb{C}\backslash D}|\nabla\chi|^2\,d^2z + \frac{1}{6\pi}\oint_{\partial D}\kappa\chi\,ds - \frac{1}{4\pi}\int_{\mathbb{C}\backslash D}\Delta\chi\,d^2z \tag{4.11}$$

    (the divergent terms proportional to the area and perimeter are omitted). In the lhs, $\det(-\Delta_{\mathbb{C}\backslash D})$ is the regularized spectral determinant of the Laplace operator in the Euclidean metric. This determinant is expressed entirely in terms of the boundary value of the metric $e^\phi\,dw\,d\bar{w} = dz\,d\bar{z}$ induced by the conformal map $w(z)$ of the exterior of D onto the exterior of the unit circle:

    $$\log\det(-\Delta_{\mathbb{C}\backslash D}) = \frac{1}{12\pi}\oint_{\partial D}\phi(\kappa + e^{-\phi})\,ds = \frac{1}{12\pi}\oint_{|w|=1}(\phi\partial_n\phi + 2\phi)|dw|. \tag{4.12}$$

    Equations (4.11), (4.12) give

    $$\log\det(-e^{-2\chi}\Delta_{\mathbb{C}\backslash D}) = -\frac{1}{12\pi}\left[\int_{|w|>1}|\nabla(\phi+\chi)|^2\,d^2w - 2\oint_{|w|=1}(\phi+\chi)|dw|\right]$$
    $$- \frac{1}{4\pi}\int_{\mathbb{C}\backslash D}\Delta\chi\,d^2z \tag{4.13}$$

- *Determinant formula (a conjecture).* As a by-product we suggest another interesting formula for the spectral determinant. Let $t_k = \frac{1}{2\pi ik}\oint_{\partial D} z^{-k}\bar{z}\,dz$ be harmonic moments of the exterior of the domain D and πt_0 be its area. Let us assume that the domain is such that all the moments at $k > m$ vanish. Then, up to a constant, the exterior spectral determinant is

    $$-12\log\det(-\Delta_{\mathbb{C}\backslash D}) = \log\det_{m\times m}\left(\frac{\partial^3 F_0}{\partial t_0\partial t_j\partial t_k}\right) - (m^2 - 3m + 3)\frac{\partial^2 F_0}{\partial t_0^2} - (m-1)\log\bar{t}_m,$$

    where F_0 is given by (4.6) with $\sigma = 1/\pi$. Although the rhs looks like a complex number, it is actually real, as will be clear from the derivation below (see (6.7), (6.8)).

4.2. Comments and details

The mean density in the leading order. In the leading order equation (3.3) reads

$$2\beta \int \log|z - \zeta| \rho_0(\zeta)\,\mathrm{d}^2\zeta + W(z) = \Lambda, \qquad \Lambda = \frac{1}{2}(2 - \beta)\hbar\lambda.$$

It must be valid in the domain where $\rho \neq 0$. We assume that this domain (the support of the density) is bounded, otherwise the normalization condition, i.e., that the integral $\int \rho\,\mathrm{d}^2 z$ be finite, can hardly be satisfied. We denote the support of the density by D. In this paper, in order to avoid additional technical complications, we consider only the case of connected domains D.

Upon taking the z-derivative of the equation above, we get, for z inside the domain D,

$$\partial\varphi_0(z) = \partial W(z) \qquad (z \in \mathsf{D}) \tag{4.14}$$

with

$$\varphi_0(z) = -\beta \int \log|z - \zeta|^2 \rho_0(\zeta)\,\mathrm{d}^2\zeta$$

being the 2D Coulomb potential created by the equilibrium ('classical') configuration of the charges characterized by the density ρ_0. This equation just states that the total force experienced by a charge at any point z, where $\rho_0(z) \neq 0$, is zero. Indeed, interaction with the other charges, $\partial\varphi_0(z)$ is compensated by the force $\partial W(z)$ due to the external field.

The solution is conveniently expressed through the function σ (4.1). For the model to be well defined, we assume that $\sigma(z) > 0$ and tends to a positive constant as $|z| \to \infty$. Applying $\partial_{\bar{z}}$ to both sides of equation (4.14), one obtains the solution $\rho_0(z) = \sigma(z)/\beta$ inside D and $\rho_0(z) = 0$ outside it. In terms of the potential φ_0 this solution reads

$$\varphi_0(z) = -\int_{\mathsf{D}} \log|z - \zeta|^2 \sigma(\zeta)\,\mathrm{d}^2\zeta. \tag{4.15}$$

Assuming, without loss of generality, that $0 \in \mathsf{D}$ and $W(0) = 0$, we fix $\Lambda = -\varphi_0(0)$, and so $W(z) = \varphi_0(z) - \varphi_0(0)$. Plugging this into (3.5), (3.6), we find the leading contributions to the free energy (4.6), (4.7). F_0 is the electrostatic energy of the domain D charged with the density $\sigma(z)$ and with a point-like compensating charge at the origin. Different aspects and applications of the functional (4.6) were discussed in [8, 24, 25, 31–34].

The most non-trivial part of the problem is to find the shape of D. It is determined by equation (4.14) and by the normalization condition. Using the Cauchy integral formula, we can write these conditions in the form

$$\begin{cases} \oint_{\partial\mathsf{D}} \frac{\partial W(\zeta)\,\mathrm{d}\zeta}{z - \zeta} = 0 & \text{for all } z \in \mathsf{D} \\ \int_{\mathsf{D}} \sigma(\zeta)\,\mathrm{d}^2\zeta = \beta t_0. \end{cases} \tag{4.16}$$

To solve them for D, provided $W(z)$ and t_0 are given, amounts to a version of the inverse potential problem in two dimensions. We assume that the potential W is such that the solution exists and is unique. In general, the solution is not available in an explicit form. In this paper we do not address this question. Our goal is to express corrections to the free energy (4.6) in terms of the domain D.

Pair correlation functions. The correlation functions in the leading order can be found using the general variational formulae (1.6), where the exact free energy is replaced by F_0:

$$\lim_{\hbar \to 0} \langle \rho(z) \rangle = \frac{\delta F_0}{\delta W(z)} = \rho_0(z), \qquad \lim_{\hbar \to 0} \langle \rho(z_1)\rho(z_2) \rangle_c = \hbar^2 \frac{\delta\rho_0(z_1)}{\delta W(z_2)}.$$

Basically, these are linear response relations used in the Coulomb gas theory [2]. In this approximation, the 2D Coulomb plasma is represented as a continuous charged fluid, so the

information about its discrete microscopic structure is lost. This is a good approximation at distances much larger than the mean distance between the charges.

Here are the results for the correlation functions of the type (1.7). The mean value is obvious from the result for the mean density given above:

$$\hbar\beta\,\langle\mathrm{Tr}f\rangle = \int_{\mathsf{D}}\sigma(z)f(z)\,\mathrm{d}^2z + O(\hbar). \tag{4.17}$$

The variation w.r.t. the potential yields the connected parts of pair correlators [24]. To present the result, we need some elements of the Dirichlet boundary value problem. Given a function $f(z)$, let $f^H(z)$ be its *harmonic continuation* from the boundary of D to its exterior, i.e., the solution of the exterior Dirichlet boundary value problem. The solution is given by the formula

$$f^H(z) = -\frac{1}{2\pi}\oint_{\partial\mathsf{D}} f(\xi)\partial_n G(z,\xi)|\mathrm{d}\xi|, \tag{4.18}$$

where $G(z,\xi)$ is the Green function of the domain $\mathbb{C}\backslash\mathsf{D}$. It is the symmetric function of two points uniquely determined by the following properties:

$$\Delta_z G(z,\zeta) = 2\pi\delta(z-\zeta) \quad \text{in } \mathbb{C}\backslash\mathsf{D}, \qquad G(z,\zeta) = 0 \qquad \text{if } z \in \partial\mathsf{D}.$$

For simply connected exterior domains $\mathbb{C}\backslash\mathsf{D}$ the Green function can be expressed through the conformal map $w(z)$ from $\mathbb{C}\backslash\mathsf{D}$ onto the exterior of the unit circle[8]:

$$G(z,\zeta) = \log\left|\frac{w(z)-w(\zeta)}{1-w(z)\overline{w(\zeta)}}\right|. \tag{4.19}$$

As $\zeta \to z$, it has the logarithmic singularity $G(z,\zeta) \to \log|z-\zeta|$. The connected pair correlator (1.7) is then given by [24]

$$4\pi\beta\,\langle\mathrm{Tr}\,f\mathrm{Tr}g\rangle_c = 4\pi\beta\int f(z)\hat{K}^{-1}(z,z')g(z')\,\mathrm{d}^2z\,\mathrm{d}^2z'$$
$$= \int_{\mathsf{D}}\nabla f\nabla g\,\mathrm{d}^2z - \oint_{\partial\mathsf{D}}f\partial_n g^H\,\mathrm{d}s, \tag{4.20}$$

where ∇f is the gradient of the function f. Alternatively, with the help of the Green formula equation (4.20) can be rewritten as a bulk integral (4.22), where the integration goes over the exterior domain and the entire plane:

$$4\pi\beta\int f(z)\hat{K}^{-1}(z,z')g(z')\,\mathrm{d}^2z\,\mathrm{d}^2z' = -\int_{\mathbb{C}\backslash\mathsf{D}}\nabla(f-f^H)\nabla(g-g^H)\,\mathrm{d}^2z + \int_{\mathbb{C}}\nabla f\nabla g\,\mathrm{d}^2z. \tag{4.21}$$

In particular, for the connected correlation functions of the fields $\varphi(z_1)$, $\varphi(z_2)$ this formula gives (if $z_{1,2} \in \mathbb{C}\backslash\mathsf{D}$):

$$\frac{1}{2\beta\hbar^2}\,\langle\varphi(z_1)\varphi(z_2)\rangle_c = G(z_1,z_2) - G(z_1,\infty) - G(\infty,z_2) - \log\frac{|z_1-z_2|}{r} + O(\hbar), \tag{4.22}$$

where $\log r = \lim_{z\to\infty}(\log|z| + G(z,\infty))$ is the (external) conformal radius of D.

Spectral determinants. The relation (4.5) between the spectral determinants,

$$\log\det\hat{K} = \log\det(-\mathrm{e}^{-2\chi}\Delta_{\mathbb{C}\backslash\mathsf{D}}) - \log\det(-\mathrm{e}^{-2\chi}\Delta_{\mathbb{C}}), \tag{4.23}$$

can be understood using formula (4.20). The presence of the bulk and boundary terms on the rhs suggests to separate boundary and bulk values of the functions f and g. Specifically, we write $f = f_H + \tilde{f}$, where f_H is the harmonic continuation of the f to the interior of D while

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[8] Throughout the paper, the map $w(z)$ is normalized as $w(z) = z/r + O(1)$ as $z \to \infty$ with a real r.

the boundary value of $\tilde{f}$ is zero, and similarly for g. Using the Green theorem, one can see that the bulk and boundary contributions completely separate[9]:

$$4\pi\beta \int f\hat{K}^{-1}g = \int_D \tilde{f}(-e^{-2\chi}\Delta)\tilde{g}\,e^{2\chi}\,d^2z + \oint_{\partial D} f\hat{N}g\,ds.$$

Here

$$\hat{N}g(z) = \partial_n^+ g_H(z) - \partial_n^- g^H(z)$$

is the Neumann jump operator ($\partial_n^{\pm}$ are normal derivatives from inside and outside, respectively) which sends a function on the boundary to the difference of normal derivatives of its harmonic continuations inside and outside the domain D. This means that the operator $\hat{K}^{-1}$ is the direct sum of the Laplace operator in D (with the Dirichlet b.c.) and the Neumann jump operator. Therefore, $\det(\hat{K}^{-1}) \simeq \det(-e^{-2\chi}\Delta_D)\det\hat{N}$, and so

$$F_1^{(0)} = \tfrac{1}{2}\log\det(-e^{2\chi}\Delta_D) + \tfrac{1}{2}\log\det\hat{N} \tag{4.24}$$

(we omit an irrelevant factor). This expression can be simplified by means of the following relation between the properly regularized functional determinants [28]:

$$\log\det(-e^{-2\chi}\Delta_D) + \log\det\hat{N} + \log\det(-e^{-2\chi}\Delta_{\mathbb{C}\setminus D}) = \log P(D) + \log\det(-e^{-2\chi}\Delta_{\mathbb{C}}).$$

Here $P(D) = \oint_{\partial D}\sqrt{\pi\sigma}\,ds$ is the perimeter of D in the metric $\pi\sigma$.

This relation, rigorously proven in [28], is known in the mathematical literature as the 'surgery formula'. It is clearly motivated by the 'cut and paste' physical arguments. Consider the free bosonic theory in the whole plane with the quadratic action $S_0 = \int |\nabla X|^2\,d^2z$ (for brevity, we do not indicate the metric explicitly). The path integral $\int[DX]\exp(-S_0[X])$ is equal to $(\det(-\Delta_{\mathbb{C}}))^{-1/2}$. On the other hand, let us fix a domain D and represent the action as

$$S_0 = \int_D |\nabla X|^2\,d^2z + \int_{\mathbb{C}\setminus D} |\nabla X|^2\,d^2z.$$

Decompose the field X inside D into the sum of the field $\tilde{X}$ such that $\tilde{X} = 0$ on ∂D and the harmonic field X_H: $X = \tilde{X} + X_H$. Let $X = \tilde{X} + X^H$ be the similar decomposition for the field X outside D (we then have $X_H = X^H$ on ∂D). It is easy to see that these fields separate in the action as follows:

$$S_0[X] = -\int_D \tilde{X}\Delta\tilde{X}\,d^2z - \int_{\mathbb{C}\setminus D} \tilde{X}\Delta\tilde{X}\,d^2z + \oint_{\partial D} X_H\hat{N}X_H\,ds.$$

This separation implies the surgery formula. The term $\log P(D)$ is due to the zero mode of the Neumann jump operator which we did not take into account.

Hence, $F_1^{(0)}$, the 'quantum' part of F_1, is to be identified with $\tfrac{1}{2}\log\det(-e^{-2\chi}\Delta_{\mathbb{C}}) - \tfrac{1}{2}\log\det(-e^{-2\chi}\Delta_{\mathbb{C}\setminus D})$, where the determinants have to be properly regularized. Some additional efforts are required to refine these arguments. In particular, one should justify the choice of the background metric in D and take care of the zero mode of the Neumann jump operator. This is beyond the scope of the present paper. The next section provides an alternative derivation of F_1.

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[9] We deliberately keep the factor $e^{2\chi} = \pi\sigma$ in some formulae in order to emphasize the fact that the Laplace–Beltrami operator is an invariant operator with respect to the metric $\pi\sigma$ written in the Weyl gauge. Although the metric cancels in the formula below, it appears in the spectral determinants.

5. Corrections to the free energy from the loop equation

In the previous sections we have found that the asymptotic expansion of the partition function as $\hbar \to 0$ has the form

$$Z_N = N! \hbar^{\frac{1}{2}(2-\beta)N} e^{\gamma N} \exp\left(\frac{F_0}{\hbar^2} + \frac{F_{1/2}}{\hbar} + F_1 + \sum_{k \geqslant 3} \hbar^{k-2} F_{k/2}\right). \tag{5.1}$$

The corresponding expansions of the mean values of ρ and φ are

$$\langle \rho(z) \rangle = \rho_0(z) + \hbar \rho_{1/2}(z) + \hbar^2 \rho_1(z) + O(\hbar^3) \tag{5.2}$$

$$\langle \varphi(z) \rangle = \varphi_0(z) + \hbar \varphi_{1/2}(z) + \hbar^2 \varphi_1(z) + O(\hbar^3), \tag{5.3}$$

where $\rho_i(z) = \delta F_i / \delta W(z)$ and $\varphi_i(z) = -\beta \int \rho_i(\zeta) \log|z - \zeta|^2 \, d^2\zeta$. The terms F_0 and $F_{1/2}$ are given by (4.6) and (4.7), respectively. The higher corrections are due to fluctuations of the charged particles around the equilibrium state. In principle, they can be found by expanding the loop equation (2.5),

$$\frac{1}{2\pi} \int \frac{\partial W(\zeta) \langle \Delta \varphi(\zeta) \rangle}{z - \zeta} \, d^2\zeta = \langle (\partial \varphi(z))^2 \rangle + (2 - \beta)\hbar \langle \partial^2 \varphi(z) \rangle, \tag{5.4}$$

in powers of $\hbar$ and solving the inhomogeneous linear integral equations obtained in this way. A similar approach has been developed in the case of Hermitian matrix ensembles [16]. This is what we are going to do in this section. We restrict ourselves by $F_{1/2}$ and F_1. Calculations of higher order corrections are rather tedious.

5.1. Iteration of the loop equation

As was already pointed out, the main contribution to the partition function as $\hbar \to 0$ comes from a configuration, where the charges are 'frozen' at their equilibrium positions. Correspondingly, the averages take their 'classical' values $\langle \varphi(z) \rangle = \varphi_0(z)$, and multipoint correlators factorize in the leading order: $\langle \partial \varphi(z) \partial \varphi(z') \rangle = \partial \varphi_0(z) \partial \varphi_0(z')$. Under this assumption, the loop equation becomes a closed relation for φ_0:

$$\frac{1}{2\pi} \int \frac{\partial W(\zeta) \Delta \varphi_0(\zeta)}{z - \zeta} \, d^2\zeta = (\partial \varphi_0(z))^2, \tag{5.5}$$

where we have omitted the last term in (5.4) which is of the next order in $\hbar$. Let us apply $\partial_{\bar{z}}$ to both sides of the equation. This yields $\partial W(z) \Delta \varphi_0(z) = \partial \varphi_0(z) \Delta \varphi_0(z)$. Since $\Delta \varphi_0(z) \propto \rho_0(z)$, we obtain

$$\rho_0(z)[\partial \varphi_0(z) - \partial W(z)] = 0. \tag{5.6}$$

This equation should be solved with the additional constraints $\int \rho_0(z) \, d^2z = t_0$ (normalization) and $\rho_0(z) \geqslant 0$ (positivity). The equation tells us that either $\partial \varphi_0(z) = \partial W(z)$ or $\rho_0(z) = 0$. Applying $\partial_{\bar{z}}$, we get $\Delta \varphi_0(z) = \Delta W(z)$. This gives the solution for ρ_0 and φ_0 already obtained in section 2 by less formal arguments (see (4.15)).

Now we are in a position to develop the $\hbar$-expansion of the loop equation (2.5). First of all, we rewrite it identically in the form

$$\frac{1}{2\pi} \int L(z, \zeta) \langle \Delta \varphi(\zeta) \rangle \, d^2\zeta = (\partial \varphi_0(z))^2 - (\partial(\langle \varphi(z) \rangle$$

$$- \varphi_0(z)))^2 - \langle (\partial[\varphi(z) - \langle \varphi(z) \rangle])^2 \rangle - (2 - \beta)\hbar \langle \partial^2 \varphi(z) \rangle,$$

which is ready for the $\hbar$-expansion. Here

$$L(z, \zeta) = \frac{\partial W(\zeta) - \partial \varphi_0(z)}{\zeta - z} \tag{5.7}$$

is the kernel of the integral operator on the lhs (the 'loop operator'). The zeroth order in $\hbar$ gives equation (5.6) which implies the familiar result $\varphi_0(z) = -\int_D \log|z - \zeta|^2 \sigma(\zeta) \, d^2\zeta$ for the φ_0. To proceed, one should insert the series (5.3) into the loop equation and separate terms of order $\hbar, \hbar^2$ etc. The terms of order $\hbar$ and $\hbar^2$ give

$$\frac{1}{2\pi} \int L(z, \zeta) \Delta \varphi_{1/2}(\zeta) \, d^2\zeta = -(2 - \beta) \partial^2 \varphi_0(z)$$
$$\frac{1}{2\pi} \int L(z, \zeta) \Delta \varphi_1(\zeta) \, d^2\zeta = -[(\partial \varphi_{1/2}(z))^2 + (2 - \beta) \partial^2 \varphi_{1/2}(z)] - \omega(z) \tag{5.8}$$

where

$$\omega(z) = \lim_{\hbar \to 0} [\hbar^{-2} \lim_{z' \to z} \langle \partial \varphi(z) \partial \varphi(z') \rangle_c] \tag{5.9}$$

is the connected part of the pair correlator at merging points.

The expansion of the loop equation can be continued order by order. In principle, this gives a recurrence procedure to determine the coefficients $\varphi_k(z)$. However, each step requires solving integral equations in the plane, which is not easy to do explicitly. Another difficulty is that in general one cannot extend these equations to the interior of the support of the density because the $\hbar$-expansion may break down or change its form there. Indeed, in the domain where the density is macroscopically nonzero, the microscopic structure of the gas becomes essential, and one needs to know correlation functions at small scales. Nevertheless, at least in the first two orders in $\hbar^2$ the equations above can be solved assuming that $z \in \mathbb{C} \backslash D$. Note that in this region all the functions $\varphi_k(z)$ are harmonic. If these functions are known, the corresponding expansion coefficients of the free energy in can be obtained by 'integration' of the variational formulae (1.6).

5.2. The solution for $F_{1/2}$

We start with the order $\hbar$. We need to solve the first equation in (5.8). Using (1.5), (1.6), we rewrite it in the form

$$\int \frac{\delta F_{1/2}}{\delta W(z)} L(a, z) \, d^2z = \frac{2 - \beta}{2\beta} \int_D \frac{\sigma(z) \, d^2z}{(a - z)^2} \tag{5.10}$$

where a is an arbitrary point in $\mathbb{C} \backslash D$. Using the variational technique developed in [24] (see also appendix A), one can verify that $F_{1/2}$ given in equation (4.7) does solve this equation.

$F_{1/2}$ from the loop equation. Here we give some details of this calculation. Exactly the same scheme is used in the next subsection while solving the loop equation for F_1.

It is convenient to use the notation (4.1) $\chi(z) = \log \sqrt{\pi \sigma(z)}$ in terms of which $F_{1/2} = -\frac{2-\beta}{\pi\beta} \int_D e^{2\chi} \chi \, d^2z$. The variation of this functional reads

$$\delta F_{1/2} = -\frac{2 - \beta}{\pi \beta} \left[\oint_{\partial D} e^{2\chi} \chi \delta n \, ds + \int_D e^{2\chi} (2\chi + 1) \delta \chi \, d^2z \right]. \tag{5.11}$$

Here

$$\delta n(z) = \frac{\partial_n (\delta W(z) - \delta W^H(z))}{4\pi \sigma(z)} \tag{5.12}$$
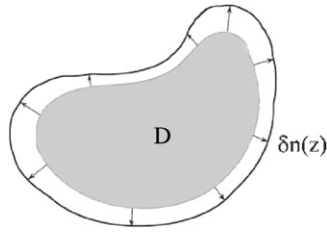
Figure 1. The normal displacement of the boundary.

is the normal displacement of the boundary (figure 1) under variation of the potential, with the convention that $\delta n > 0$ for outward displacement (for the proof see [24, 25]) and

$$\delta\chi(z) = -\frac{\Delta\delta W(z)}{8\pi\sigma(z)}. \tag{5.13}$$

Note that the lhs of (5.10) has the meaning of the variation of $F_{1/2}$ under a small change of the potential proportional to $L(a, z)$, $\delta W(z) \propto L(a, z)$ (a plays the role of a parameter). Therefore, the result is given by (5.11) where δn and $\delta\chi$ are taken from (5.12), (5.13) with $\delta W(z) = L(a, z)$. It remains to plug the explicit form of the $L(a, z)$ (5.7) and simplify the result. In the course of this calculation, the frequently used formulae are

$$\Delta_z L(a, z) = 4\pi\,\partial_z\left(\frac{\sigma(z)}{a-z}\right) \tag{5.14}$$

and

$$\oint_{\partial D} f(z)\partial_n(L(a, z) - L^H(a, z))\,\mathrm{d}s = 2\pi\mathrm{i}\oint_{\partial D}\frac{f(z)\sigma(z)}{a-z}\,\mathrm{d}\bar{z} \tag{5.15}$$

(for any smooth function f). It is implied that $z \in D$, $a \in \mathbb{C}\backslash D$. In these formulae, the Laplace operator and the harmonic continuation are applied to z. We have

$$\int\frac{\delta F_{1/2}}{\delta W(z)}L(a, z)\,\mathrm{d}^2 z$$
$$= -\frac{2-\beta}{4\pi\beta}\left[\oint_{\partial D}\chi(z)\partial_n(L(a, z) - L^H(a, z))\,\mathrm{d}s - \frac{1}{2}\int_D(2\chi(z) + 1)\Delta L(a, z)\,\mathrm{d}^2 z\right]$$
$$= -\frac{2-\beta}{4\pi\beta}\left[-\mathrm{i}\oint_{\partial D}\frac{\chi(z)\sigma(z)}{z-a}\,\mathrm{d}\bar{z} + \int_D(2\chi(z) + 1)\partial_z\left(\frac{\sigma(z)}{z-a}\right)\,\mathrm{d}^2 z\right].$$

After transforming the first (contour) integral to the integral over the domain D,

$$\oint_{\partial D}\frac{\chi(z)\sigma(z)}{z-a}\,\mathrm{d}\bar{z} = -2\mathrm{i}\int_D\partial_z\left(\frac{\chi(z)\sigma(z)}{z-a}\right)\,\mathrm{d}^2 z,$$

one can see that the result is indeed equal to the rhs of (5.10).

The results for $\rho_{1/2}$ and $\varphi_{1/2}$. The corrections to the mean values of ρ and φ are conveniently expressed through the function χ (4.1) and its harmonic continuation $\chi^H(z)$ from the boundary of D to its exterior. In terms of these functions

$$\rho_{1/2}(z) = \frac{\delta F_{1/2}}{\delta W(z)} = \frac{2-\beta}{4\pi\beta}\left(\Theta(z; \mathsf{D})\Delta\chi(z) - \delta(z; \partial\mathsf{D})\partial_n(\chi(z) - \chi^H(z)) - \frac{1}{2}\delta'(z; \partial\mathsf{D})\right). \tag{5.16}$$

Here $\Theta(z; \mathsf{D})$ is the characteristic function of the domain D (1 if $z \in \mathsf{D}$ and 0 otherwise), $\delta(z; \partial\mathsf{D})$ is the δ-function with the support on the boundary ($\int f(z)\delta(z; \partial\mathsf{D})\,\mathrm{d}^2 z = \oint_{\partial\mathsf{D}} f(z)|\mathrm{d}z|$

for any smooth function f) and $\delta'(z; \partial \mathsf{D})$ is its 'normal derivative', i.e., a function such that $\int f(z)\delta'(z; \partial \mathsf{D})\,\mathrm{d}^2 z = -\oint_{\partial \mathsf{D}} \partial_n f(z)|\mathrm{d}z|$. The singular function $\rho_{1/2}$ is to be understood as being integrated with any smooth test function. The first term in the rhs describes the change of density in the bulk. The second one (the 'simple layer') describes a shift of the boundary of the domain D. The last one, the 'double layer', means smoothing off the edge of the density support (for $\beta \neq 2$). Using the operator $\hat{K}$ introduced in section 3, the correction $\rho_{1/2}$ can be written as

$$\rho_{1/2}(z) = -(2 - \beta)[\hat{K}^{-1}\chi](z) - \frac{2 - \beta}{8\pi\beta}\delta'(z; \partial \mathsf{D}).$$

At $\beta = 2$, the correction $\rho_{1/2}$ vanishes. The first correction to $\langle \varphi(z) \rangle$ reads

$$\varphi_{1/2}(z) = \begin{cases} -(2 - \beta)\left[\chi(z) - \chi^H(\infty) + \frac{1}{2}\right], & z \in \mathsf{D} \\ -(2 - \beta)\left[\chi^H(z) - \chi^H(\infty)\right], & z \in \mathbb{C}\backslash\mathsf{D}. \end{cases} \tag{5.17}$$

Due to the double layer, this function is discontinuous across the boundary. Therefore, for $z \in \mathbb{C}\backslash\mathsf{D}$ we have $\partial\varphi_{1/2}(z) = -(2 - \beta)\partial\chi^H(z)$.

5.3. The solution for F_1

Now we have to solve the second equation in (5.8), where $\partial\varphi_{1/2} = -(2 - \beta)\partial\chi^H$ and $\omega(z)$ is still to be found. If the point z is in $\mathbb{C}\backslash\mathsf{D}$, then equation (4.22) yields [24]

$$\langle\partial\varphi(z)\partial\varphi(z')\rangle_c = 2\beta\hbar^2\partial_z\partial_{z'}(G(z, z') - \log|z - z'|) + O(\hbar^3)).$$

Since the rhs is regular for all $z, z' \in \mathbb{C}\backslash\mathsf{D}$, the points can be merged without any difficulty and the result does not depend on the particular limit $z' \to z$. Using the expression of the Green function (4.19) through the conformal map $w(z)$, we obtain

$$12 \lim_{z' \to z} \partial_z\partial_{z'}(G(z, z') - \log|z - z'|) = \frac{w'''(z)}{w'(z)} - \frac{3}{2}\left(\frac{w''(z)}{w'(z)}\right)^2 = \{w; z\},$$

where we use the standard notation for the Schwarzian derivative. The function ω is thus given by

$$\omega(z) = \frac{\beta}{6}\{w; z\}. \tag{5.18}$$

After plugging the above result for $\varphi_{1/2}$, the second equation of (5.8) acquires the form:

$$\int \frac{\delta F_1}{\delta W(\zeta)} L(z, \zeta)\,\mathrm{d}^2\zeta = \alpha^2[(\partial\chi^H(z))^2 - \partial^2\chi^H(z)] + \frac{1}{12}\{w; z\}, \tag{5.19}$$

where $\alpha = \sqrt{\frac{2}{\beta}} - \sqrt{\frac{\beta}{2}}$ (as is in (4.1)). We consider this equation for $z \in \mathbb{C}\backslash\mathsf{D}$, where both the sides are harmonic functions.

Our strategy is as follows. We make a guess for F_1 and then show that the variational derivative obeys (5.19). More precisely, let I^{trial} be a trial functional of D (and thus of W) represented as an integral over the domain D or its boundary. The form of the trial functionals is suggested by the path integral arguments of the previous section. Since F_1 has dimension 0, we consider only dimensionless functionals. We want to find $\int \delta I^{\mathrm{trial}}/\delta W(z)L(a, z)\,\mathrm{d}^2 z$ and compare with the rhs of (5.19) (again, a is a point in $\mathbb{C}\backslash\mathsf{D}$). The latter quantity can be computed by the method outlined in the previous subsection.

A comment on the meaning of the calculations below is in order. They consist of recognizing that the lhs of equation (5.19) is a variation of F_1 over the holomorphic component of the metric and then applying the Polyakov–Alvarez formula (4.11). We plan to elaborate on this point elsewhere.

The structure of equation (5.19) suggests to find different terms of the solution separately. Let us start with the term proportional to α^2. Here are the main steps of the calculations. Consider the functional

$$I^{(1)} = \frac{1}{4\pi} \left(\int_D |\nabla\chi|^2 \, \mathrm{d}^2 z - \oint_{\partial D} \chi \, \partial_n \chi^H \, \mathrm{d}s \right). \tag{5.20}$$

Its variation is (see appendix B)

$$\delta I^{(1)} = \frac{1}{4\pi} \oint_{\partial D} [\partial_n(\chi - \chi^H)]^2 \delta n \, \mathrm{d}s + \frac{1}{2\pi} \oint_{\partial D} \delta\chi \, \partial_n(\chi - \chi^H) \, \mathrm{d}s - \frac{1}{2\pi} \int_D \delta\chi \, \Delta\chi \, \mathrm{d}^2 z. \tag{5.21}$$

In the same way as for $F_{1/2}$ one can check that

$$\int \frac{\delta I^{(1)}}{\delta W(z)} L(a, z) \, \mathrm{d}^2 z = (\partial\chi^H(a))^2 - \partial^2\chi^H(a);$$

so F_1 is expected to contain the term $\alpha^2 I^{(1)}$ (compare with $F_1^{(1)}$ given by (4.9)). This contribution is of the 'classical' nature since no fluctuations of the charges positions are taken into account.

Next, we consider

$$I^{(2)} = -\frac{1}{2\pi} \oint_{|w|=1} (\phi \, \partial_n \phi + 2\phi) |\mathrm{d}w| \tag{5.22}$$

(we remind that $\phi(w) = \log|z'(w)|$ where $z(w)$ is the conformal map from the exterior of the unit circle onto $\mathbb{C}\backslash D$ inverse to the $w(z)$). The variation of this functional is found in appendix B (equation (B.5)):

$$\delta I^{(2)} = \frac{1}{2\pi} \oint (\nu^2(z)\{w; z\} + \overline{\nu^2(z)\{w; z\}} - 2\kappa^2(z)) \delta n(z) \, \mathrm{d}s. \tag{5.23}$$

Here

$$\nu(z) = |w'(z)| \frac{w(z)}{w'(z)} \tag{5.24}$$

is the normal unit vector to the boundary[10] and

$$\kappa(z) = \partial_n \log \left| \frac{w(z)}{w'(z)} \right|$$

is the local curvature of the boundary (counted w.r.t. the outward pointing normal vector). Using the rules explained above, we get

$$\int \frac{\delta I^{(2)}}{\delta W(z)} L(a, z) \, \mathrm{d}^2 z = -\frac{1}{4\pi\mathrm{i}} \oint_{\partial D} \frac{\nu^2\{w; z\} + \overline{\nu^2\{w; z\}} - 2\kappa^2}{a - z} \, \mathrm{d}\bar{z}.$$

Now consider the functional

$$I^{(3)} = \frac{1}{2\pi} \int_D |\nabla\chi|^2 \, \mathrm{d}^2 z + \frac{1}{\pi} \oint_{\partial D} \kappa\chi \, \mathrm{d}s \tag{5.25}$$

with the variation

$$\delta I^{(3)} = \frac{1}{2\pi} \oint_{\partial D} \left(|\nabla\chi|^2 - 2\partial_s^2 \chi + 2\kappa \, \partial_n \chi \right) \delta n \, \mathrm{d}s + \frac{1}{\pi} \oint_{\partial D} (\kappa + \partial_n \chi) \delta\chi \, \mathrm{d}s - \frac{1}{\pi} \int_D \Delta\chi \, \delta\chi \, \mathrm{d}^2 z.$$

[10] It is worthwhile to mention here the useful formulae for normal and tangential derivatives, $\partial_n f = \nu \partial_z f + \bar{\nu} \partial_{\bar{z}} f$, $\partial_s f = \mathrm{i}\nu \partial_z f - \mathrm{i}\bar{\nu} \partial_{\bar{z}} f$, and for the line element on the boundary curve, $\mathrm{d}s = \frac{\mathrm{d}z}{\mathrm{i}\nu(z)} = \mathrm{i}\nu(z) \, \mathrm{d}\bar{z}$.

The variational derivative $\delta I^{(3)}/\delta W$ looks rather complicated. However, for its convolution with $L(a, z)$ one obtains a surprisingly simple result:

$$\int \frac{\delta I^{(3)}}{\delta W(z)} L(a, z) \, \mathrm{d}^2 z = -\frac{1}{2\pi} \oint_{\partial D} \frac{\kappa \, \mathrm{d} s}{(a - z)^2} = \frac{1}{2\pi \mathrm{i}} \oint_{\partial D} \frac{\kappa^2 + \mathrm{i}\kappa'}{a - z} \, \mathrm{d}\bar{z}$$

(note that χ cancels!). Combining it with the corresponding result for $I^{(2)}$, we obtain

$$\int \frac{\delta(I^{(2)} - I^{(3)})}{\delta W(z)} L(a, z) \, \mathrm{d}^2 z = -\frac{1}{4\pi \mathrm{i}} \oint_{\partial D} \frac{v^2\{w; z\} + \overline{v^2\{w; z\}} + 2\mathrm{i}\kappa'}{a - z} \, \mathrm{d}\bar{z}$$

$$= -\frac{1}{2\pi \mathrm{i}} \oint_{\partial D} \frac{v^2(z)\{w; z\}}{a - z} \, \mathrm{d}\bar{z} = \frac{1}{2\pi \mathrm{i}} \oint_{\partial D} \frac{\{w; z\}}{a - z} \, \mathrm{d} z = \{w(a); a\}$$

where (A.2) has been used.

At last, the functional

$$I^{(0)} = \frac{1}{2\pi} \int_D \Delta\chi \, \mathrm{d}^2 z \tag{5.26}$$

with the variation

$$\delta I^{(0)} = \frac{1}{2\pi} \oint_{\partial D} (\Delta\chi \delta n + \partial_n(\delta\chi)) \, \mathrm{d} s$$

is a 'zero mode' of the loop operator:

$$\int \frac{\delta I^{(0)}}{\delta W(z)} L(a, z) \, \mathrm{d}^2 z = 0.$$

Summing all the contributions with appropriate coefficients that follow from the rhs of the loop equation, we thus find F_1:

$$F_1 = F_1^{(1)} + F_1^{(0)} = \alpha^2 I^{(1)} + \tfrac{1}{12}(I^{(2)} - I^{(3)}) + \mu I^{(0)},$$

where the coefficient μ cannot be determined from the loop equation restricted to $\mathbb{C}\backslash D$. (In the whole plane, the loop operator does not have zero modes but we have to be restricted to $\mathbb{C}\backslash D$ because of unknown properties of correlation functions at small distances in the bulk.) For $\beta = 1$, the coefficient μ can be fixed by comparison with the explicit solution for centrosymmetric potentials (see appendix C):

$$\mu = -\tfrac{1}{8} \qquad (\text{at } \beta = 1).$$

With the help of the Green theorem, we present the result in the form appearing in section 4:

$$F_1 = \frac{1}{24\pi} \left[\int_{|w|>1} |\nabla(\phi + \chi)|^2 \, \mathrm{d}^2 w - 2 \oint_{|w|=1} (\phi + \chi)|\mathrm{d} w| \right]$$

$$+ \frac{\alpha^2}{4\pi} \left[\int_D |\nabla\chi|^2 \, \mathrm{d}^2 z - \oint_{\partial D} \chi \partial_n \chi^H \, \mathrm{d} s \right]$$

$$+ \frac{\mu}{2\pi} \int_D \Delta\chi \, \mathrm{d}^2 z - \frac{1}{24\pi} \int_{\mathbb{C}} |\nabla\chi|^2 \, \mathrm{d}^2 z, \tag{5.27}$$

where χ in the first two terms is regarded as a function of w via $\chi = \chi(z(w))$. Let us also list some equivalent forms:

$$F_1 = -\frac{1}{24\pi} \oint_{|w|=1} (\phi \partial_n \phi + 2\phi)|\mathrm{d} w| - \frac{1 - 6\alpha^2}{24\pi} \left[\int_D |\nabla\chi|^2 \, \mathrm{d}^2 z + 2 \oint_{\partial D} \kappa \chi \, \mathrm{d} s \right]$$

$$- \frac{\alpha^2}{4\pi} \oint_{\partial D} \chi (\partial_n \chi^H + 2\kappa) \, \mathrm{d} s + \frac{\mu}{2\pi} \oint_{\partial D} \partial_n \chi \, \mathrm{d} s, \tag{5.28}$$

$$F_1 = \frac{1 - 6\alpha^2}{24\pi} \left[\int_{|w|>1} |\nabla(\phi + \chi)|^2 \, d^2w - 2 \oint_{|w|=1} (\phi + \chi)|dw| - \int_{\mathbb{C}} |\nabla(\phi + \chi)|^2 \, d^2z \right]$$
$$+ \frac{\alpha^2}{4\pi} \left[\int_{|w|>1} |\nabla(\phi + \chi^H)|^2 \, d^2w - 2 \oint_{|w|=1} (\phi + \chi^H)|dw| \right] + \frac{\mu}{2\pi} \int_{\mathsf{D}} \Delta\chi \, d^2z. \tag{5.29}$$

6. Models with quasiharmonic potentials

The models with W of the form

$$W = -|z|^2 + V(z) + \overline{V(z)}$$

('quasiharmonic potentials') generalize the Ginibre–Girko ensemble [15]. Note that for potentials of this form the integral (1.1) diverges unless V is quadratic or logarithmic with suitable coefficients. The simplest way to give sense to the integral when it diverges at infinity is to introduce a cutoff, i.e., integrate over a suitably chosen big but finite domain in the plane. Then the large-N expansion is well defined and the results for the general potential presented above still make sense. For details and rigorous proofs, see [35].

In the case of quasiharmonic potentials the formula for F_1 drastically simplifies since the function χ vanishes, and so only the first integral in (5.28) survives:

$$F_1 = -\frac{1}{24\pi} \oint_{|w|=1} (\phi \partial_n \phi + 2\phi)|dw|. \tag{6.1}$$

In the particular case $\beta = 1$, $W(z) = -z\bar{z}$ the formula yields $F_1 = -\frac{1}{12} \log t_0$, which coincides with the result of [5] obtained by a direct calculation.

According to the conjecture of section 4.1, equation (6.1) can be understood as the formula for the regularized determinant of the Laplace operator $\Delta_{\mathbb{C}\backslash\mathsf{D}} = 4\partial_z\partial_{\bar{z}}$ in the exterior domain $\mathbb{C}\backslash\mathsf{D}$ with the Dirichlet boundary conditions:

$$F_1 = -\frac{1}{2} \log \det(-\Delta_{\mathbb{C}\backslash\mathsf{D}}). \tag{6.2}$$

The first term is the bulk contribution (for the metric induced by the conformal map it reduces to a boundary integral), while the second term is a net boundary term. The 'classical' contribution $F_1^{(1)}$ to F_1 vanishes in this case.

6.1. The case of rational $\partial V(z)$ (quadrature domains)

There is a special class of domains for which our result (6.1) can be made more explicit. Consider domains such that $z'(w)$ is a rational function,

$$z'(w) = r \prod_{i=0}^{m-1} \frac{w - a_i}{w - b_i}.$$

In the mathematical literature, they are called quadrature domains [36]. One can show that quadrature domains are density supports for the models with potentials such that $\partial V(z)$ is a rational function. All the points a_i and b_i must be inside the unit circle, otherwise the map $z(w)$ is not conformal. As $w \to \infty$, $z(w)$ can be represented as a Laurent series of the form $z(w) = rw + u_0 + O(w^{-1})$. On the unit circle we have $|dw| = \frac{dw}{iw}$ and $\phi(w) = \frac{1}{2}(\log z'(w) + \log \bar{z}'(w^{-1}))$, where the first and the second term (the Schwarz reflection)

are analytic outside and inside it, respectively. (Recall that $\bar{z}(w) \equiv \overline{z(\bar{w})}$ and our notation $\bar{z}'(w^{-1})$ means $d\bar{z}(u)/du$ at the point $u = w^{-1}$.) Plugging this into (6.1), we get

$$
F_1 = -\frac{1}{24\pi i} \oint_{|w|=1} \log z'(w) \left[\frac{1}{2} \partial_w \log z'(w) + \frac{1}{w} \right] dw
$$
$$
- \frac{1}{24\pi i} \oint_{|w|=1} \log \bar{z}'(w^{-1}) \frac{dw}{w} - \frac{1}{48\pi i} \oint_{|w|=1} \log \bar{z}'(w^{-1}) \frac{z''(w)}{z'(w)} \, dw.
$$

The integrals can be calculated by taking residues either outside or inside the unit circle. The poles are at ∞, at 0, and at the points a_i and b_i. The result is

$$
F_1 = -\frac{1}{24} \left(\log r^4 + \sum_{z'(a_i)=0} \log \bar{z}'\left(a_i^{-1}\right) - \sum_{z'(b_i)=\infty} \log \bar{z}'\left(b_i^{-1}\right) \right). \tag{6.3}
$$

If the potential $V(z)$ is polynomial, $V(z) = \sum_{k=1}^m t_k z^k$, i.e., $t_k = 0$ as $k > m$ for some $m > 0$, then the series for the conformal map $z(w)$ truncates: $z(w) = rw + \sum_{l=0}^{m-1} u_l w^{-l}$ and

$$
z'(w) = r \prod_{i=0}^{m-1} (1 - a_i w^{-1})
$$

is a polynomial in w^{-1} (all poles b_i of $z'(w)$ merge at the origin). Then the last sum in (6.3) becomes $m \log r$ and the formula (6.3) gives

$$
F_1 = -\frac{1}{24} \log \left(r^4 \prod_{z'(a_j)=0} \frac{\bar{z}'\left(a_j^{-1}\right)}{r} \right) = -\frac{1}{24} \log \left(r^4 \prod_{i,j=0}^{m-1} (1 - \bar{a}_i a_j) \right). \tag{6.4}
$$

This formula is essentially identical to the genus-1 correction to the free energy of the Hermitian 2-matrix model with a polynomial potential computed in [20].

6.2. The determinant formula

For polynomial $V(z) F_1$ enjoys an interesting determinant representation. Set

$$
D_m := \det \left(\frac{\partial^3 F_0}{\partial t_0 \partial t_j \partial t_k} \right)_{0 \leqslant j,k \leqslant m-1}
$$

where F_0 is the leading contribution to the free energy regarded as a function of t_0 and the coefficients t_k (and the complex conjugate coefficients $\bar{t}_k$). We need the *residue formula* for the third-order derivatives of F_0 [37]:

$$
\frac{\partial^3 F_0}{\partial t_j \partial t_k \partial t_l} = \frac{1}{2\pi i} \oint_{|w|=1} \frac{h_j(w) h_k(w) h_l(w)}{z'(w) \bar{z}'(w^{-1})} \frac{dw}{w}. \tag{6.5}
$$

Here $h_j(w)$ is the following polynomial in w of degree j:

$$
h_j(w) = w \frac{d}{dw} [(z^j(w))_+] \qquad \text{for} \quad j \geqslant 1 \qquad \text{and} \qquad h_0(w) = 1,
$$

where $(\cdots)_+$ is the positive degree part of the Laurent series. Using this formula, we compute

$$
D_m = \frac{1}{(2\pi i)^m} \oint_{|w_0|=1} \frac{dw_0}{w_0} \cdots \oint_{|w_{m-1}|=1} \frac{dw_{m-1}}{w_{m-1}} \frac{\det[h_j(w_j) h_k(w_j)]}{\prod_{l=0}^{m-1} z'(w_l) \bar{z}'\left(w_l^{-1}\right)}. \tag{6.6}
$$

Clearly, the determinant in the numerator can be substituted by $\frac{1}{m} \det^2(h_j(w_k))$ and $\det[h_j(w_k)] = (m-1)! r^{\frac{1}{2}m(m-1)} \Delta_m(w_i)$, where $\Delta_m(w_i)$ is the Vandermonde determinant.

Each integral in (6.6) is given by the sum of residues at the points a_i inside the unit circle (the residues at $w_i = 0$ vanish). Computing the residues and summing over all permutations of the points a_i, we get

$$D_m = (-1)^{\frac{1}{2}m(m-1)}((m-1)!)^2 r^{m(m-3)} \frac{\prod_j a_j^{m-1}}{\prod_{i,j}(1 - a_j \bar{a}_i)}. \tag{6.7}$$

It is not difficult to see that $\prod_{i=1}^m a_i = (-1)^m m(m-1)r^{m-2}\bar{t}_m$ (we regard t_m, the last nonzero coefficient of the $V(z)$, as a fixed parameter) and so we represent F_1 (6.4) in the form

$$F_1 = \tfrac{1}{24}\log D_m - \tfrac{1}{12}(m^2 - 3m + 3)\log r - \tfrac{1}{24}(m-1)\log \bar{t}_m + \text{const}, \tag{6.8}$$

where const is a numerical constant. Since $\partial_{t_0}^2 F_0 = 2\log r$ (see [24, 25]), we obtain that F_1, for models with polynomial potentials of degree m, is expressed through derivatives of F_0.

According to the conjecture of section 4.1, the formula below suggests a new representation of the spectral determinant of the Laplace operator. Up to a constant we have

$$-\frac{1}{2}\log\det(-\Delta_{\mathbb{C}\backslash\mathsf{D}}) = \frac{1}{24}\log\det_{m\times m}\left(\frac{\partial^3 F_0}{\partial t_0 \partial t_j \partial t_k}\right)$$
$$- \frac{1}{24}(m^2 - 3m + 3)\frac{\partial^2 F_0}{\partial t_0^2} - \frac{1}{24}(m-1)\log \bar{t}_m \tag{6.9}$$

where j, k run from 0 to $m-1$. Similar determinant formulae are known for genus-1 corrections to free energy in topological field theories [38]. However, they have not been identified with spectral determinants.

7. Spectral determinant of the Laplace–Beltrami operator for the Dirichlet problem

To give an interpretation of the result for F_1, we recall the formula for the spectral determinant of the Laplace–Beltrami operator in a compact bounded domain M in the plane (assumed to have topology of a disc). For the derivation, see [29, 30, 39, section 1] and appendix D. The Laplace operator acts on functions vanishing on the boundary (the Dirichlet boundary conditions). Being written in the Weyl gauge, $g_{ab} = e^{2\Phi}\delta_{ab}$, it has the form

$$\frac{1}{\sqrt{g}}\partial_a(\sqrt{g}g^{ab}\partial_b) = 4\,e^{-2\Phi(w)}\partial_w\partial_{\bar{w}} = e^{-2\Phi}\Delta_\mathsf{M},$$

where w is a holomorphic coordinate on M. The Polyakov–Alvarez formula gives the difference between the spectral determinants of the Laplace operators in the metric $e^{2\Phi}\,dw\,d\bar{w}$ and in some fixed reference metric. Assuming that the latter is just the standard flat metric $dw\,d\bar{w}$ in the plane, the formula reads (see (4.42) in the first paper in [30])

$$\log\det(-e^{-2\Phi}\Delta_\mathsf{M}) = -\frac{1}{4\pi\epsilon^2}\int_\mathsf{M} e^{2\Phi}\,d^2w + \frac{1}{4\sqrt{\pi}\epsilon}\oint_{\partial\mathsf{M}} e^\Phi|dw| + \frac{1}{6}\log\epsilon$$
$$- \frac{1}{12\pi}\int_\mathsf{M}|\nabla\Phi|^2\,d^2w - \frac{1}{6\pi}\oint_{\partial\mathsf{M}}\hat{\kappa}\Phi|dw| - \frac{1}{4\pi}\int_\mathsf{M}\Delta\Phi\,d^2w. \tag{7.1}$$

Here ϵ is an ultraviolet cutoff and $\hat{\kappa}$ is the curvature of the boundary w.r.t. the reference metric. The first three terms diverge as $\epsilon \to 0$. A sketch of the derivation is given in appendix D.

We are going to show that our result for the 'quantum' part of the free energy, $F_1^{(0)}$, given by (4.8), (5.27)–(5.29), agrees with (7.1) generalized to exterior domains. Indeed, let us adopt this formula for the exterior of the domain D. For this purpose, we map it to the exterior of

the unit circle U_{ext} by the conformal map $z \to w(z)$ and choose M in (7.1) to be the exterior of the unit circle, $\mathsf{M} = \mathsf{U}_{\text{ext}}$, with the metric $\Phi(w) = \chi(z(w)) + \phi(w)$. Then

$$e^{2\Phi} \, dw \, d\bar{w} = e^{2\chi} \, dz \, d\bar{z}. \tag{7.2}$$

Taking into account that for the exterior of the unit circle $\hat{\kappa} = -1$, we rewrite (7.1) as

$$\log \det(-e^{-2\chi} \Delta_{\mathbb{C}\backslash\mathsf{D}}) = -\frac{1}{4\epsilon^2} \int_{\mathbb{C}\backslash\mathsf{D}} \sigma \, d^2 z + \frac{1}{4\epsilon} \oint_{\partial\mathsf{D}} \sqrt{\sigma} |dz| + \frac{1}{6} \log \epsilon$$
$$- \frac{1}{12\pi} \left[\int_{|w|>1} |\nabla\Phi|^2 \, d^2 w - 2 \oint_{|w|=1} \Phi |dw| \right] - \frac{1}{4\pi} \int_{|w|>1} \Delta\Phi \, d^2 w. \tag{7.3}$$

On the complex plane $\mathbb{C}$ (without boundary) with coordinate z in the metric $e^{2\chi}$ we have

$$\log \det(-e^{-2\chi} \Delta_{\mathbb{C}}) = -\frac{1}{4\epsilon^2} \int_{\mathbb{C}} \sigma \, d^2 z + \frac{1}{3} \log \epsilon - \frac{1}{12\pi} \int_{\mathbb{C}} |\nabla\chi|^2 \, d^2 z. \tag{7.4}$$

Therefore, we can write

$$\log \frac{\det(-e^{-2\chi} \Delta_{\mathbb{C}})}{\det(-e^{-2\chi} \Delta_{\mathbb{C}\backslash\mathsf{D}})} = -\frac{t_0}{4\epsilon^2} + \frac{1}{6} \log \epsilon - \frac{P(\mathsf{D})}{4\sqrt{\pi}\epsilon}$$
$$+ \frac{1}{12\pi} \left[\int_{|w|>1} |\nabla\Phi|^2 \, d^2 w - 2 \oint_{|w|=1} \Phi |dw| - 3 \oint_{|w|=1} \partial_n \chi |dw| \right]$$
$$- \frac{1}{12\pi} \int_{\mathbb{C}} |\nabla\chi|^2 \, d^2 z. \tag{7.5}$$

The first two divergent terms can be absorbed by normalization and so we ignore them in what follows. The third one is proportional to the perimeter of the density support (in the metric $e^{2\chi}$):

$$P(\mathsf{D}) = \oint_{\partial\mathsf{D}} \sqrt{\pi\sigma} \, ds.$$

It can be directly verified that the perimeter functional obeys the relation

$$\int \frac{\delta P(\mathsf{D})}{\delta W(z)} L(a, z) \, d^2 z = 0 \qquad (a \in \mathbb{C}\backslash\mathsf{D});$$

so all the three divergent terms are 'zero modes' of the loop operator (for the first two this is obvious).

Comparing (7.5) with (5.27), we can represent F_1 in the form

$$F_1 = \frac{1}{2} \log \frac{\det'(-e^{-2\chi} \Delta_{\mathbb{C}})}{\det'(-e^{-2\chi} \Delta_{\mathbb{C}\backslash\mathsf{D}})} + \frac{\alpha^2}{4\pi} \left[\int_{\mathsf{D}} |\nabla\chi|^2 \, d^2 z - \oint_{\partial\mathsf{D}} \chi \partial_n \chi^H \, ds \right] + \frac{\mu + \frac{1}{4}}{2\pi} \oint_{\partial\mathsf{D}} \partial_n \chi \, ds. \tag{7.6}$$

Here $\log \det'$ means the finite (as $\epsilon \to 0$) part of (7.5). It is the 'quantum' part of the answer. Obviously, F_1 is related to gravitational anomalies. This relation awaits further clarification.

Acknowledgments

Appendix A. Some useful formulae

Here we fix the notation and collect some formulae often used in the main text.

Integral formulae. Throughout the paper, all contours are assumed to be anticlockwise oriented and the normal vector looks outward D.

- Cauchy's integral formula (f is any smooth function):

$$\frac{1}{2\pi i} \oint_{\partial \mathsf{D}} \frac{f(\zeta)\, d\zeta}{z - \zeta} - \frac{1}{\pi} \int_{\mathsf{D}} \frac{\partial_{\bar\zeta} f(\zeta)\, d^2\zeta}{z - \zeta} = \begin{cases} -f(z), & z \in \mathsf{D} \\ 0, & z \in \mathbb{C}\backslash\mathsf{D}. \end{cases}$$

- The Green formula: $\int_{\mathsf{D}} f \Delta g\, d^2 z = -\int_{\mathsf{D}} \nabla f \nabla g\, d^2 z + \oint_{\partial \mathsf{D}} f \partial_n g |dz|$.
- The Hadamard variational formula: the variation of the Green function of $\mathbb{C}\backslash\mathsf{D}$ under small deformation of the domain D with the normal displacement $\delta n(z)$ is

$$\delta G(z_1, z_2) = \frac{1}{2\pi} \oint_{\partial \mathsf{D}} \partial_n G(z_1, \xi) \partial_n G(z_2, \xi) \delta n(\xi) |d\xi|.$$

Variation of contour integrals. Consider the contour integral of the general form:

$$\oint_{\partial \mathsf{D}} F(f(z), \partial_n f(z))\, ds,$$

where F is any fixed function. Calculating the linear response to the deformation of the contour (described by the normal displacement $\delta n(z)$), one should vary all items in the integral independently and add the results. There are four elements to be varied: the support of the integral $\oint$, the ∂_n, the line element ds and the function f. By variation of the $\oint$ we mean integration of the old function over the new contour. This gives $\oint \delta n \partial_n F\, ds$. The change of the slope of the normal vector results in $\delta(\partial_n) = -\partial_s(\delta n)\partial_s$. The rescaling of the line element gives $\delta\, ds = \kappa \delta n\, ds$, where $\kappa(z)$ is the local curvature of the boundary curve. At last, we have to vary the function f if it explicitly depends on the contour. In particular, if this function is the harmonic extension of a contour-independent function on the plane, its variation on the boundary is given by

$$\delta f^H(z) = \partial_n(f(z) - f^H(z))\delta n(z), \qquad z \in \partial \mathsf{D}.$$

This is an equivalent form of the Hadamard variational formula. (For more details, see, e.g., [24, 32], where the Hadamard formula is extensively used.)

The curvature. The local curvature of the boundary curve is defined as $\kappa = d\theta/ds$, where θ is the angle between the outward pointing normal vector to the curve and the x-axis. The formula

$$\kappa(z) = \partial_n \log \left| \frac{w(z)}{w'(z)} \right| \tag{A.1}$$

is an immediate consequence of the definition. For practical calculations, we also need the Laplace operator at boundary points in terms of normal (∂_n) and tangential (∂_s) derivatives,

$$\Delta = \partial_n^2 + \partial_s^2 + \kappa \partial_n,$$

and the formula for the tangential derivative of the curvature, $\kappa' = \partial_s \kappa$, through the boundary value of the Schwarzian derivative $\{w; z\} = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2$:

$$\kappa'(z) = \mathcal{I}m(\nu^2(z)\{w; z\}) \tag{A.2}$$

(ν is the complex unit normal vector (5.24)). The variation of the curvature under small deformations of the contour is

$$\delta\kappa(z) = -\left(\partial_s^2 + \kappa^2(z)\right)\delta n(z). \tag{A.3}$$

Indeed, $\kappa + \delta\kappa = \frac{d(\theta + \delta\theta)}{ds + \delta\,ds}$, where $\delta\theta = \partial_s(\delta n)$ is the change of the slope of the normal vector. The line element is rescaled as $ds \rightarrow ds + \kappa\,\delta n\,ds$, so $\delta\,ds = \kappa\,\delta n\,ds$. In the first order in δn we then have $\kappa + \delta\kappa = \kappa - \partial_s^2(\delta n) - \kappa^2\delta n$, that is, (A.3).

Appendix B. Variations of the trial functionals

Let us consider in detail the most complicated case of the functional

$$I^{(2)} = -\frac{1}{2\pi}\oint_{|w|=1}(\phi\partial_n\phi + 2\phi)|dw|.$$

To vary this functional, it is convenient to pass to the z-plane and introduce the function

$$q(z) = \log\left|\frac{w'(z)}{w(z)}\right|. \tag{B.1}$$

It is harmonic in $\mathbb{C}\backslash\mathsf{D}$ with the logarithmic singularity at infinity: $q(z) = -\log|z| + O(1/|z|)$ as $z \rightarrow \infty$. Obviously, $q^H(z) = \log|w'(z)| = q(z) + \log|w(z)|$. In terms of this function, the curvature is $\kappa(z) = -\partial_n q(z)$, and

$$I^{(2)} = \frac{1}{2\pi}\oint_{\partial\mathsf{D}}(-q\,\partial_n q + q\,\mathrm{e}^q)\,ds. \tag{B.2}$$

We would like to find the linear response of this quantity to small deformations of the contour.

For clarity, let us deal with the two terms in (B.2) separately. We apply the rules given above and get, after some cancellations,

$$\delta\oint q\,\partial_n q\,ds = \oint \delta n|\nabla q|^2\,ds + \oint(\delta q\,\partial_n q + q\,\partial_n\delta q)\,ds$$

where $|\nabla q|^2 = (\partial_n q)^2 + (\partial_s q)^2$. Using the Green theorem and the behaviour of the function q at infinity, one can see that the contributions of the two terms in the second integral are the same. The variation δq at the boundary can be found by means of the Hadamard formula. The result is

$$\delta q = -\mathrm{e}^{-q}\partial_n(\mathrm{e}^q\delta n)^H. \tag{B.3}$$

Therefore,

$$\delta\oint q\,\partial_n q\,ds = \oint[|\nabla q|^2 - 2\,\mathrm{e}^q\partial_n(\mathrm{e}^{-q}\partial_n q)^H]\delta n\,ds$$

where we used the Green formula again. The harmonic continuation is achieved by means of the identity $\mathrm{e}^{-q}\partial_n q = 2\mathcal{R}e\left(\frac{w}{w'}\partial_z q\right)$, whose rhs is explicitly harmonic and thus provides the desired harmonic continuation. Next, a straightforward calculation shows that

$$\mathrm{e}^q\partial_n(\mathrm{e}^{-q}\partial_n q)^H = -\mathrm{e}^q\partial_s(\mathrm{e}^{-q}\partial_s q) = (\partial_s q)^2 - \partial_s^2 q$$

and we get

$$\delta\oint q\,\partial_n q\,ds = \oint\left(2\partial_s^2 q - |\nabla q|^2 + 2\kappa^2\right)\delta n\,ds.$$

The combination $2\partial_s^2 q - |\nabla q|^2$ can be transformed as follows:

$$2\partial_s^2 q - |\nabla q|^2 = -2\left[v^2(z)\left(\partial_z^2 q - (\partial_z q)^2\right) + \mathrm{c.\,c.}\right],$$

where $2\left(\partial_z^2 q - (\partial_z q)^2\right) = \{w; z\} + \frac{1}{2}\left(\frac{w'}{w}\right)^2$ and $\{w; z\}$ is the Schwarzian derivative. Combining these formulae, we get

$$2\partial_s^2 q - |\nabla q|^2 = -(v^2\{w; z\} + \overline{v^2\{w; z\}} + |w'|^2). \tag{B.4}$$

Now to the second term in (B.2). We have

$$
\delta \oint q \, \mathrm{e}^q \, \mathrm{d}s = \oint \delta n \partial_n q \, \mathrm{e}^q + \oint (\mathrm{e}^q \delta q + q \, \mathrm{e}^q \delta q) \, \mathrm{d}s
$$

$$
= \oint \delta n \, \mathrm{e}^q \partial_n (q - q^H) \, \mathrm{d}s = - \oint \delta n \, \mathrm{e}^{2q} \, \mathrm{d}s,
$$

where we use the identity $\partial_n \log|w(z)| = |w'(z)|$ and take into account that $|w'| = \mathrm{e}^q$ at the boundary. Combining the results, we obtain

$$
\delta I^{(2)} = \frac{1}{2\pi} \oint (v^2(z)\{w; z\} + \overline{v^2(z)\{w; z\}} - 2\kappa^2(z))\delta n(z) \, \mathrm{d}s. \tag{B.5}
$$

Variations of other trial functionals go in a similar way. Here we just list the results:

$$
\delta \int_{\mathsf{D}} |\nabla \chi|^2 \, \mathrm{d}^2 z = \oint_{\partial \mathsf{D}} |\nabla \chi|^2 \delta n \, \mathrm{d}s + 2 \int_{\mathsf{D}} \nabla \chi \nabla (\delta \chi) \, \mathrm{d}^2 z \, \delta \oint_{\partial \mathsf{D}} \kappa \chi \, \mathrm{d}s
$$

$$
= \oint_{\partial \mathsf{D}} \left(-\partial_s^2 \chi + \kappa \partial_n \chi \right) \delta n \, \mathrm{d}s + \oint_{\partial \mathsf{D}} \kappa \delta \chi \, \mathrm{d}s \, \delta \oint_{\partial \mathsf{D}} \chi \partial_n \chi^H \, \mathrm{d}s
$$

$$
= \oint_{\partial \mathsf{D}} (|\nabla \chi|^2 - (\partial_n (\chi - \chi^H))^2) \delta n \, \mathrm{d}s + 2 \oint_{\partial \mathsf{D}} \delta \chi \partial_n \chi^H \, \mathrm{d}s \, \delta \int_{\mathsf{D}} \Delta \chi \, \mathrm{d}^2 z
$$

$$
= \oint_{\partial \mathsf{D}} \Delta \chi \delta n \, \mathrm{d}s + \oint_{\partial \mathsf{D}} \partial_n (\delta \chi) \, \mathrm{d}s.
$$

To express the right-hand sides through the variation of the potential W, one should plug the formulae (5.12), (5.13).

Appendix C. The centrosymmetric potential at $\beta = 1$

In the centrosymmetric case the potential W does not depend on the angular coordinate in the plane. We set $W(z) = W_{\mathrm{rad}}(|z|^2)$. At $\beta = 1$, the orthogonal polynomials technique is applicable. The symmetry of the potential implies that the orthogonal polynomials are simply monomials z^n. Therefore, the expression for the partition function simplifies considerably:

$$
Z_N = \prod_{n=0}^{N-1} h_n
$$

where

$$
h_n = \int_{\mathbf{C}} |z|^{2n} \, \mathrm{e}^{\frac{1}{\hbar} W} \, \mathrm{d}^2 z = \pi \int_0^\infty x^n \, \mathrm{e}^{\frac{1}{\hbar} W_{\mathrm{rad}}(x)} \, \mathrm{d}x.
$$

Introduce the function

$$
h(t) = \int_0^\infty \mathrm{e}^{\frac{1}{\hbar}(W_{\mathrm{rad}}(x) + t \log x)} \, \mathrm{d}x, \tag{C.1}
$$

then

$$
\log Z_N = N \log \pi + \sum_{n=0}^{N-1} \log h(n\hbar). \tag{C.2}
$$

We are going to find the $\hbar$-expansion of $\log Z_N$ using the $\hbar$-expansion of $h(t)$ obtained by the saddle point method and the Euler–MacLaurin formula. The necessary formulae are collected below.

$\hbar$-expansion of the free energy: a direct calculation. First of all, we note that the density support D is an axially symmetric domain. We assume that it is a disc (not a ring) of radius R.

The density of eigenvalues is $\sigma(z) = \sigma_{\rm rad}(|z|^2)$. Clearly, $\pi\sigma_{\rm rad}(x) = -\partial_x(x\partial_x W_{\rm rad}(x))$. The radius R is a function of t_0 defined by the relation

$$t_0 = -R^2 W'_{\rm rad}(R^2) \tag{C.3}$$

(here and below prime means $f'(x) = {\rm d}f/{\rm d}x$). This relation follows directly from the definition

$$t_0 = -\frac{1}{4\pi}\int_{\rm D} \Delta W\,{\rm d}^2 z = -\frac{1}{4\pi}\oint_{\partial {\rm D}} \partial_n W\,{\rm d}s$$

taking into account that $\partial_n W = 2R W'_{\rm rad}(R^2)$.

The saddle point x_c in (C.1) is determined by the equation

$$t = -x_c W'_{\rm rad}(x_c). \tag{C.4}$$

Note the similarity with (C.3): x_c is the squared radius of the disc for the filling $n = t/\hbar$. We assume that there is only one saddle point. The standard technique (see (C.8)) yields the following asymptotic expansion as $\hbar \to 0$:

$$h(t) = e^{\frac{1}{\hbar}(W_{\rm rad}(x_c) - W'_{\rm rad}(x_c)x_c \log x_c)}\sqrt{\frac{2\hbar x_c}{\sigma_{\rm rad}(x_c)}}(1 + \hbar p_1 + O(\hbar^2)), \tag{C.5}$$

where p_1 is given by (C.9) with $S(x) = W_{\rm rad}(x) + t\log x$. It is easy to calculate

$$S''(x_c) = -\frac{\pi\sigma_{\rm rad}(x_c)}{x_c}$$

$$S'''(x_c) = -\frac{\pi\sigma'_{\rm rad}(x_c)}{x_c} + 2\frac{\pi\sigma_{\rm rad}(x_c)}{x_c^2}$$

$$S^{IV}(x_c) = -\frac{\pi\sigma''_{\rm rad}(x_c)}{x_c} + 3\frac{\pi\sigma'_{\rm rad}(x_c)}{x_c^2} - 6\frac{\pi\sigma_{\rm rad}(x_c)}{x_c^3}.$$

Plugging this stuff into (C.9), we get

$$p_1 = \frac{1}{24\pi\sigma_{\rm rad}(x_c)}\left(5x_c\left(\frac{\sigma'_{\rm rad}(x_c)}{\sigma_{\rm rad}(x_c)}\right)^2 - 11\frac{\sigma'_{\rm rad}(x_c)}{\sigma_{\rm rad}(x_c)} - 3x_c\frac{\sigma''_{\rm rad}(x_c)}{\sigma_{\rm rad}(x_c)} + \frac{2}{x_c}\right).$$

In terms of the function $\chi(z) = \chi_{\rm rad}(|z|^2) = \frac{1}{2}\log(\pi\sigma_{\rm rad})$ this expression acquires the form

$$p_1 = \frac{1}{\pi\sigma_{\rm rad}(x_c)}\left(\frac{1}{3}x_c(\chi'_{\rm rad}(x_c))^2 - \frac{1}{4}\partial_x(x\partial_x\chi_{\rm rad}(x))\Big|_{x=x_c} - \frac{2}{3}\chi'_{\rm rad}(x_c) + \frac{1}{12x_c}\right).$$

The Euler–MacLaurin formula applied to (C.2) yields

$$\log Z_N = N\log\pi + \frac{1}{\hbar}\int_0^{t_0}\log h(t)\,{\rm d}t - \frac{1}{2}\log h(t_0) + \frac{\hbar}{12}\partial_t\log h(t)\Big|_{t=t_0} + C,$$

where C is a constant which does not depend on t_0. (It can be found by a more detailed analysis around the point $t = 0$.) From (C.5) we have

$$\log h(t) = \frac{1}{\hbar}(W_{\rm rad}(x_c) - W'_{\rm rad}(x_c)x_c\log x_c) + \log\sqrt{2\pi\hbar}$$
$$+ \frac{1}{2}\log x_c - \chi_{\rm rad}(x_c) + \hbar p_1 + O(\hbar^2). \tag{C.6}$$

The integral over t is transformed into the integral over x_c using ${\rm d}t/{\rm d}x_c = \pi\sigma_{\rm rad}(x_c)$:

$$\int_0^{t_0}\log h(t)\,{\rm d}t = \pi\int_0^{R^2}\sigma_{\rm rad}(x)\log h(t(x))\,{\rm d}x.$$

Finally, we obtain

$$F_0 = \pi \int_0^{R^2} (W_{\mathrm{rad}}(x) - W'_{\mathrm{rad}}(x)x \log x)\sigma_{\mathrm{rad}}(x) \, \mathrm{d}x$$

$$F_{1/2} = -\frac{\pi}{2} \int_0^{R^2} \sigma_{\mathrm{rad}}(x) \log(\pi \sigma_{\mathrm{rad}}(x)) \, \mathrm{d}x \tag{C.7}$$

$$F_1 = -\frac{1}{12} \log R^2 - \frac{1}{6}\chi_{\mathrm{rad}}(R^2) - \frac{1}{4}R^2\chi'_{\mathrm{rad}}(R^2) + \frac{1}{3} \int_0^{R^2} x(\chi'_{\mathrm{rad}}(x))^2 \, \mathrm{d}x.$$

Reducing the 2D integrals in (5.27) to 1D integrals or values of χ_{rad}, χ'_{rad} at the boundary,

$$\int_{\mathsf{D}} |\nabla \chi|^2 \, \mathrm{d}^2z = 4\pi \int_0^{R^2} x(\chi'_{\mathrm{rad}}(x))^2 \, \mathrm{d}x$$

$$\oint_{\partial \mathsf{D}} \kappa \chi \, \mathrm{d}s = 2\pi \chi_{\mathrm{rad}}(R^2)$$

$$\int_{\mathsf{D}} \Delta \chi \, \mathrm{d}^2z = 4\pi R^2 \chi'_{\mathrm{rad}}(R^2),$$

and taking into account that $\partial_n \chi^H = 0$, we have

$$F_1 = -\frac{1}{12} \log R^2 - \frac{1}{6}\chi_{\mathrm{rad}}(R^2) + \frac{1}{3} \int_0^{R^2} x(\chi'_{\mathrm{rad}}(x))^2 \, \mathrm{d}x + 2\mu R^2 \chi'_{\mathrm{rad}}(R^2).$$

Comparing with (C.7), we conclude that $\mu = -1/8$.

Asymptotic formulae. Evaluating the integral $\int e^{\frac{1}{\hbar} S(x)} \, \mathrm{d}x$ by the saddle point method around the critical point x_c, $S'(x_c) = 0$, we get the asymptotic expansion as $\hbar \to 0$:

$$\int e^{\frac{1}{\hbar} S(x)} \, \mathrm{d}x = e^{\frac{1}{\hbar} S(x_c)} \sqrt{\frac{2\pi\hbar}{|S''(x_c)|}} (1 + \hbar p_1 + \hbar^2 p_2 + \cdots), \tag{C.8}$$

where

$$p_n = \frac{\Gamma\left(n + \frac{1}{2}\right)}{\sqrt{2\pi}(2n)!} |S''(x_c)| \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{2n} \left[-\frac{S(x) - S(x_c)}{(x - x_c)^2}\right]^{-n-\frac{1}{2}} \Bigg|_{x=x_c}.$$

In particular,

$$p_1 = \frac{5(S'''(x_c))^2 + 3|S''(x_c)|S^{IV}(x_c))}{24|S''(x_c)|^3}. \tag{C.9}$$

In the text we also need the Euler–MacLaurin formula

$$\sum_{n=0}^{N-1} f(n) = \int_0^N f(x) \, \mathrm{d}x - \frac{1}{2}(f(N) - f(0)) + \frac{1}{12}(f'(N) - f'(0)) + \cdots.$$

Appendix D. Derivation of the Polyakov–Alvarez formula

In this appendix we outline the derivation of the formula (4.11) for the spectral determinant of the Laplace operator. We use the notation introduced in the main text.

Consider a free Bose field X defined in a planar (compact) domain B and vanishing on its boundary. For simplicity, we assume that B has topology of a disc. Let $e^{2\chi} \, \mathrm{d}z \, \mathrm{d}\bar{z}$ be a background conformal metric on B. The classical action,

$$S = \frac{1}{4\pi} \int_{\mathsf{B}} X(-e^{-2\chi}\Delta)X e^{2\chi} \, \mathrm{d}^2z, \tag{D.1}$$

does not depend on the metric but the quantum theory does. The log of partition function of this field $\mathcal{F} = -\frac{1}{2}\log\det(-e^{-2\chi}\Delta_B)$ represents the spectral determinant of the Laplace–Beltrami operator $e^{-2\chi}\Delta_B = 4\,e^{-2\chi(z)}\partial_z\partial_{\bar{z}}$ acting on functions vanishing on the boundary (the Dirichlet b.c.). Let $z(w)$ be the univalent conformal map from the unit disc U onto B and $\phi = \log|dz/dw|$. This map induces the conformal metric $e^{2\Phi}\,dw\,d\bar{w}$ on U in the holomorphic coordinate w, with the conformal factor

$$e^{2\Phi} = e^{2\chi}|z'(w)|^2 = e^{2(\chi+\phi)} = \sqrt{g}.$$

The Laplace–Beltrami operator in the coordinate w is $4\,e^{-2\Phi(w)}\partial_w\partial_{\bar{w}}$. The spectral determinant of this operator can be expressed in terms of Φ and the curvature of the boundary. The result is referred to as the Polyakov–Alvarez formula [29, 30]. Below we give a short derivation of this formula.

Variation of the partition function over the metric introduces the trace $T_{z\bar{z}} = \langle|\partial X|^2\rangle$ and the holomorphic component $T = \langle(\partial X)^2\rangle$ of the stress energy tensor:

$$-\frac{1}{\pi}T_{z\bar{z}} = \frac{1}{2}\left(\frac{\delta}{\delta\chi} + \frac{\delta}{\delta\phi}\right)\mathcal{F} = \sqrt{g}\,\frac{\delta\mathcal{F}}{\delta\sqrt{g}}. \tag{D.2}$$

The trace of the s.e.t. has bulk and boundary parts, and so has $\mathcal{F}$:

$$T_{z\bar{z}}\,dz\,d\bar{z} = T_{z\bar{z}}^{\text{bulk}}\,dz\,d\bar{z} + T_{z\bar{z}}^{\text{boundary}}\,ds, \qquad \mathcal{F} = \mathcal{F}^{\text{bulk}} + \mathcal{F}^{\text{boundary}}. \tag{D.3}$$

The holomorphic component is continuous across the boundary in a regular fashion. The components $T_{z\bar{z}}$ and T are related by the conservation law

$$\partial(\sqrt{g}T_{z\bar{z}}) + \sqrt{g}\bar{\partial}T = 0 \quad \text{(in the bulk)} \tag{D.4}$$

$$2\partial_s T_{z\bar{z}}^{\text{boundary}} + \mathcal{I}m(\nu^2 T) = 0 \quad \text{(on the boundary)}, \tag{D.5}$$

where ν is the unit normal vector to the boundary.

The strategy to compute $\mathcal{F}$ is as follows. First we compute the holomorphic component T, then find $T_{z\bar{z}}$ through the conservation laws (D.4), (D.5). Finally we integrate equation (D.2).

The simplest way to proceed is as follows. Since the holomorphic component is continuous across the boundary, it depends only on the overall metric $\sqrt{g} = e^{2\chi+2\phi}$. Therefore, it is sufficient to compute T at $\chi = 0$. In the coordinate w, the domain is the unit disc, so $T(w)(dw)^2$ vanishes. The conformal transformation $w \to z$ generates the metric $e^{-2\phi}$ and transforms the holomorphic component of the stress energy tensor as $T \to T + \frac{1}{12}\{w; z\}$. Therefore[11],

$$T(z) = \tfrac{1}{12}\{w; z\} = -\tfrac{1}{24}[(\partial\log|dz/dw|^2)^2 + 2\partial^2\log|dz/dw|^2]$$

at $\chi = 0$, or, more generally,

$$T = -\tfrac{1}{24}((\partial\log\sqrt{g})^2 + 2\partial^2\log\sqrt{g}) \tag{D.6}$$

at $\chi \neq 0$. The next step is to find the trace of the stress energy tensor. The bulk part,

$$T_{z\bar{z}}^{\text{bulk}} = \tfrac{1}{48}\Delta\log\sqrt{g} = \tfrac{1}{24}\Delta\chi, \tag{D.7}$$

follows from (D.4) and represents the gravitational anomaly. It is proportional to the scalar curvature of the metric in the bulk and does not depend on the shape of the domain. This gives (after integrating equation (D.2)) the textbook result for the bulk part of the spectral determinant:

$$\mathcal{F}^{\text{bulk}} = -\frac{1}{96\pi}\int_U \log\sqrt{g}\,\Delta\log\sqrt{g}\,d^2w. \tag{D.8}$$

---

[11] One can obtain this textbook result by the direct computation of $\langle\partial X(z)\partial X(z')\rangle \propto \partial_z\partial_{z'}G(z, z')$ and extracting the finite part of the result as $z' \to z$ with the help of the explicit formula for the Green function (4.19).

We will use the same trick in order to compute the boundary part. Let us first find it at $\chi = 0$. Comparing (D.5) and (A.2) we conclude that it is $\frac{1}{24}\partial_s\kappa$, where κ is the local curvature of the boundary. This gives the boundary part of the gravitational anomaly: $T_{z\bar{z}}^{\text{boundary}} = -\frac{1}{24}\kappa$. At $\chi \neq 0$ one should simply add the boundary curvature generated by the metric $e^{2\chi}$ to the local curvature κ:

$$T_{z\bar{z}}^{\text{boundary}}\,\mathrm{d}s = -\tfrac{1}{24}(\kappa + \partial_n\chi)\,\mathrm{d}s. \tag{D.9}$$

Using the formula $\kappa\,\mathrm{d}s = (\partial_n\phi + e^{-\phi})\,\mathrm{d}s = \partial_n\phi\,\mathrm{d}s + |\mathrm{d}w|$ (equivalent to (A.1)), the boundary contribution to the gravitational anomaly can be written as

$$T_{z\bar{z}}^{\text{boundary}}\,\mathrm{d}s = -\tfrac{1}{48}(2|\mathrm{d}w| + \partial_n\log\sqrt{g}\,\mathrm{d}s).$$

This form is ready for integration with the result

$$\mathcal{F}^{\text{boundary}} = \frac{1}{96\pi}\oint_{\partial\mathsf{U}}\log\sqrt{g}\partial_n\log\sqrt{g}|\mathrm{d}w| + \frac{1}{24\pi}\oint_{\partial\mathsf{U}}\log\sqrt{g}|\mathrm{d}w| + \frac{1}{16\pi}\oint_{\partial\mathsf{U}}\partial_n\log\sqrt{g}|\mathrm{d}w|, \tag{D.10}$$

where we take into account that ∂_n on $\partial\mathsf{D}$ is $e^{-\phi}\partial_n$ on the unit circle. Combining with (D.8), we obtain the finite part of (4.11) (recall that $\log\sqrt{g} = 2\Phi$).

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